

NECESSITY OF VANISHING SHADOW PRICE IN INFINITE HORIZON CONTROL PROBLEMS

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This paper investigates the necessary optimality conditions for uniformly overtaking optimal control on infinite horizon in the free end case. In the papers of S.M. Aseev, A.V. Kryazhinskii, V.M. Veliov, K.O. Besov there was suggested the boundary condition for equations of the Pontryagin Maximum Principle. Each optimal process corresponds to a unique solution satisfying the boundary condition. Following A.Seierstad's idea, in this paper we prove a more general geometric variety of that boundary condition. We show that this condition is necessary for uniformly overtaking optimal control on infinite horizon in the free end case. A number of assumptions under which this condition selects a unique Lagrange multiplier is obtained. The results are applicable to general non-stationary systems and the optimal objective value is not necessarily finite. Some examples are discussed.

Keywords Optimal control · infinite horizon problem · transversality condition for infinity · necessary conditions · uniformly overtaking optimal control · shadow price · unique Lagrange multiplier

Mathematics Subject Classification (2000) 49K15 · 49J45 · 37N40 · 91B62

Introduction

The Pontryagin Maximum Principle for infinite horizon problems had already been formulated in the monograph [36]; the general Maximum Principle for infinite interval was proved in [28], but such Maximum Principle has no transversality condition, and in general, selects a too broad family of extremal trajectories. A significant number [28, 6, 11, 30, 34, 42, 40, 13] of such conditions was proposed; however, as it was noted in, for example, [28, 34, 41], [6, Sect. 6], [38, Example 10.2], these conditions, as pointed out in [39], “may frequently fail to hold (conditions securing these properties may fail.) Even if they do hold, for example when strong enough growth conditions hold, these condition may fail to give any information determining the integration constants arising when integrating the adjoint equation.”

Since the necessity of this condition does not imply its nontriviality on solutions of the relations of the Maximum Principle, it is reasonable to find a condition that would select a single solution of the relations of the

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Maximum Principle for any optimal control. For this purpose, [38] proposes to find ψ^0 such that it is a pointwise limit of a sequence of shadow prices equal to zero on certain sequence $\tau \uparrow \infty$ of times. Under assumptions of [38, Theorem 6.1], such ψ^0 is unique; in what follows, it will be referred to as τ -vanishing shadow price.

In papers [4,5,6,7], Aseev and Kryazhinskii proposed the analytic expression for the shadow prices. This version of the normal form of the Maximum Principle holds with the explicitly specified shadow price. This gives a complete set of necessary optimality conditions (see [4,5,6,7,8,10]); moreover, under assumptions of [8,10,11,38], the solution of this form of the Maximum Principle is uniquely determined by the optimal control.

This paper aims to merge these two approaches, to find assumptions such that a τ -vanishing Lagrange multiplier of the Maximum Principle corresponds to every optimal control, and to express its shadow price explicitly in the form of an improper integral that depends only on optimal control and trajectory.

In this paper, we consider only the problem with free right end. It is assumed a priori that an optimal control (uniformly weakly overtaking optimal control) exists (for discussion of existence, refer to, for example, [12,14,15,16,22]). In addition to this, all functions are assumed to be smooth in x . We also do not concern ourselves with sufficient optimality conditions (see, in this connection, for example, [14,37,40,44]). Papers [39], [6, §13] actually describe sufficient conditions of optimality for same shadow price under sufficiently strong growth conditions.

The structure of the paper is as follows: We begin with formulating the general control problem and stating general notation and main assumptions (Section 1). Then, we formulate certain useful propositions from topology and stability theory (Section 2) which are later used mostly in proofs; these propositions are proved in Appendix. After that we discuss the relations of the Maximum Principle and introduce the notion of τ -vanishing Lagrange multipliers. Then we show that its existence is the necessary optimality condition (Theorem 2). Connection between τ -vanishing Lagrange multiplier and degenerate problems is investigated in Subsection 4.2; for information on the connection with the condition $\psi^0(t) \rightarrow 0$ refer to Subsection 4.1. The problems with monotonic right-hand side are investigated in Subsection 4.3. Section 5 is mainly aimed at obtaining the most diverse sets of conditions under which a τ -vanishing shadow price can be explicitly expressed via a Cauchy-type formula. Here we also discuss connections with the results of [6,8,10,38].

The last section is completely devoted to analysis of examples. We show how the choice of a sequence of τ from a number of uniformly weakly optimal solutions selects what is needed most with the help of τ -vanishing shadow price (Example 2). Example 3 demonstrates that finding the τ -vanishing Lagrange multiplier allows to solve abnormal problems in almost the same way as normal problems are solved. Example 4 shows how hard it is to determine a τ -vanishing Lagrange multiplier in cyclic problems even if we know that the optimal control is unique. In Example 5, the search for an optimal solution is reduced to a boundary value problem.

A part of results of this paper was announced in [31],[32].

1 Preliminaries

We consider the time interval $\mathbb{T} \triangleq \mathbb{R}_{\geq 0}$. The phase space of the control system is the finite-dimensional metric space $\mathbb{X} \triangleq \mathbb{R}^m$; denote the unit ball in \mathbb{X} by \mathbb{D} . Denote by \mathbb{L} the linear space of all real $m \times m$ matrices; equip \mathbb{L}

with the operator norm. The symbol E (which may be equipped with some indices) denotes various auxiliary finite-dimensional Euclidean spaces.

For a subset A of a topological space, denote by $cl A$ the closure of A , and by $int A$ the interior of A .

Slightly simplifying the notation when passing from the sequence $\tau \triangleq (\tau_n)_{n \in \mathbb{N}}$ to its subsequence τ' , we will plainly write “subsequence $\tau' \subset \tau$ ”

Let $C(T, E)$ and $C_{loc}(T, E)$ be topological spaces of all continuous functions of T to E . Let us equip the first one with extended norm $\|\cdot\|_C$ of uniform convergence. The second one is equipped the compact-open topology.

Here and below, for each integrable function a of time, the integral $\int_0^\infty a(t)dt$ is the limit $\int_0^T a(t)dt$ as $T \rightarrow \infty$. An improper integral, for example, over $[T, \infty)$, is interpreted in the same sense.

Let us also consider a finite-dimensional Euclidean space \mathbb{U} and map U of T to set of all subset of \mathbb{U} . The set \mathfrak{U} of admissible controls is understood as the set of all Borel measurable locally bounded selectors of the multi-valued map U . The topology on \mathfrak{U} is defined through the inclusion $\mathfrak{U} \subset \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{U})$.

A function $a : \mathbb{T} \times E_1 \times \mathbb{U} \rightarrow E_2$ is said to

- 1) satisfy the Carathéodory conditions if a) the function $a(\cdot, x, u) : \mathbb{T} \rightarrow E_2$ is Borel measurable for all $(x, u) \in E_1 \times \mathbb{U}$, b) the function $a(t, \cdot, \cdot) : E_1 \times \mathbb{U} \rightarrow E_2$ is continuous for a.a. $t \in \mathbb{T}$.
- 2) be locally Lipschitz continuous if for each compact subset K of $E_1 \times \mathbb{U}$ there exists a function $L_K^a \in \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{T})$ satisfying $\|a(t, x, u) - a(t, y, u)\|_{E_2} \leq L_K^a(t) \|x - y\|_{E_1}$ for all $(x, u), (y, u) \in K$, $t \in \mathbb{T}$.
- 3) be integrally bounded (on each compact subset) if for each compact subset K of $E_1 \times \mathbb{U}$ there exists a function $M_K^a \in \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{T})$ satisfying $\|a(t, x, u)\|_{E_2} \leq M_K^a(t)$ for all $(x, u) \in K$, $t \in \mathbb{T}$.

We assume the following conditions hold:

Condition (u) : U is a compact-valued map, and its graph is Borel set.

Condition (fg) : Locally Lipschitz continuous on x Carathéodory functions $f : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{X}$, $g : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{R}$, $\frac{\partial f}{\partial x} : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{L}$, $\frac{\partial g}{\partial x} : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \rightarrow \mathbb{X}$ are integrally bounded (on each compact subset); in addition, f satisfies the sublinear growth condition (see, for example, [45, 1.4.4]).

Let us consider the control system

$$\dot{x} = f(t, x, u), \quad x(0) = x_{**}, \quad t \in \mathbb{T}, \quad x \in \mathbb{X}, \quad u(t) \in U(t), \quad (1a)$$

where $x_{**} \in \mathbb{X}$ is an initial value. Now we can assign the solution of (1a) to each $u \in \mathfrak{U}$. The solution is unique and it can be extended to the whole \mathbb{T} . Let us denote it by x^u . The map $u \mapsto x^u$ of \mathfrak{U} to $C_{loc}(\mathbb{T}, \mathbb{X})$ is continuous [45].

In what follows, we study the problem of maximizing the objective integral functional

$$J^u(T) \overset{T \rightarrow \infty}{\rightsquigarrow} \max; \quad J^u(T) \triangleq \int_0^T g(t, x^u(t), u(t)) dt. \quad (1b)$$

If there is no limit in (1b), the optimality may be defined in diverse ways (for details, see [14, 16, 43, 44]); generally, we will use the following definition:

Definition 1 We say that a control $u^0 \in \mathfrak{U}$ is *weakly uniformly overtaking optimal* (see [15]) if

$$\limsup_{t \rightarrow \infty} \sup_{u \in \mathfrak{U}} (J^u(t) - J^{u^0}(t)) = 0.$$

For every sequence $\tau \triangleq (\tau_n)_{n \in \mathbb{N}} \uparrow \infty$ of times, we say that a control $u^0 \in \mathfrak{U}$ is τ -optimal if

$$\limsup_{n \rightarrow \infty} \sup_{u \in \mathfrak{U}} (J^u(\tau_n) - J^{u^0}(\tau_n)) = 0.$$

We also assume:

Condition (τ) : there exists a weakly uniformly overtaking optimal control $u^0 \in \mathfrak{U}$ for problem (1a)–(1b).

By this condition there exist an unbounded sequence $\tau = (\tau_n)_{n \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$ and some sequence $(\gamma_n)_{n \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, converging to zero, such that

$$J^{u^0}(\tau_n) \geq J^u(\tau_n) - \gamma_n^2 \quad \forall u \in \mathfrak{U}, n \in \mathbb{N}. \quad (2)$$

Then the control u^0 is τ -optimal. Fix a sequence τ . Also denote by x^0 the trajectory that corresponds to u^0 .

Thus, any weakly uniformly overtaking optimal control is τ -optimal for some sequence $\tau \uparrow \infty$. Similarly, any uniformly overtaking [15, 29] optimal control is τ -optimal for every sequence $\tau \uparrow \infty$. Since the definition of τ -optimality refines these definitions, it is especially convenient if such sequence τ is given initially.

2 Auxiliary results

2.1 The set $\tilde{\mathfrak{U}}$ of generalized controls

For each $u \in \mathbb{U}$, the symbol $\tilde{\delta}(u)$ denotes the probability measure concentrated at the point u . Denote by $\tilde{\mathfrak{U}}_n$ the family of all weakly measurable mappings η of $[0, n]$ to the set of Radon probability measures over \mathbb{U} such that $\eta(U(t)) = 1$ for a.a. $t \in [0, n]$. Let us equip this set with the topology of *-weak convergence. Then, the obtained topological space is a compact [46, IV.3.11], and the set $\mathfrak{U}_n \triangleq \{u|_{[0, n]} \mid u \in \mathfrak{U}\}$ is everywhere densely included in $\tilde{\mathfrak{U}}_n$ [46, IV.3.10] by the map $u \rightarrow \tilde{\delta} \circ u$. We also keep the notation $\tilde{u}^0 \triangleq \tilde{\delta} \circ u^0$.

Now, let us introduce the set of all maps η of \mathbb{T} into the set of Radon probability measures over \mathbb{U} such that $\eta|_{[0, n]} \in \tilde{\mathfrak{U}}_n$ for every $n \in \mathbb{N}$, and let us denote it by $\tilde{\mathfrak{U}}$. For every $n \in \mathbb{N}$, let the projections $\tilde{\pi}_n : \tilde{\mathfrak{U}} \rightarrow \tilde{\mathfrak{U}}_n$ be given by $\tilde{\pi}_n(\eta) \triangleq \eta|_{[0, n]}$ for all $\eta \in \tilde{\mathfrak{U}}$. Let us equip $\tilde{\mathfrak{U}}$ with the weakest topology such that all projections are continuous. The set $\tilde{\mathfrak{U}}$ is called the set of generalized controls.

Let us assume that for the Euclidean space E , the function $a : \mathbb{T} \times E \times \mathbb{U} \rightarrow E$ is a locally Lipschitz continuous integrally bounded Carathéodory function that satisfies the extendability condition on \mathbb{T} (for example, if the sublinear growth condition holds; see [45, 1.4.3]).

Let us fix a set $\Xi \subset E$ of initial values and the system for $u \in \mathfrak{U}$:

$$\dot{y} = a(t, y(t), u(t)), \quad y(0) = \xi \in \Xi, \quad t \in \mathbb{T}, u \in \mathfrak{U}. \quad (3a)$$

It can also be generalized for $\eta \in \tilde{\mathfrak{U}}$:

$$\dot{y} = \int_{U(t)} a(t, y(t), u) d\eta(t), \quad y(0) \in \Xi, \quad t \in \mathbb{T}, \eta \in \tilde{\mathfrak{U}}. \quad (3b)$$

Each its local solution can be extended onto the whole \mathbb{T} . For every $\eta \in \tilde{\mathfrak{U}}$, let us denote the family of all solutions $y \in C_{loc}(\mathbb{T}, E)$ of system (3b) by $\tilde{\mathfrak{A}}[\eta]$. Such transition from a system defined for $u \in \mathfrak{U}$ (like (3a)) to a generalized system, which is defined for $\eta \in \tilde{\mathfrak{U}}$ (like (3b)), will be done sufficiently often; to avoid writing

the generalized relation, we will write the initial one with the sign “ \sim .” For example, we will write $(3a)$ instead of $(3b)$. In particular, for a solution $x^\eta \in C_{loc}(\mathbb{T}, \mathbb{X})$ of the Cauchy problem $(1a)$, the function $T \mapsto \tilde{J}^\eta(T)$ could be introduced, for every $\eta \in \tilde{\mathfrak{U}}$, by the rule $(1b)$.

Proposition 1 *Assume (u). Then,*

- 1) *the space $\tilde{\mathfrak{U}}$ is a compact, and $\tilde{\delta}(\mathfrak{U})$ is everywhere dense in it;*
- 2) *the map $\tilde{\mathfrak{A}} : \tilde{\mathfrak{U}} \rightarrow C_{loc}(\mathbb{T}, E)$ is continuous and $\tilde{\mathfrak{A}}[\tilde{\delta} \circ \mathfrak{U}]$ is everywhere dense in a compact $\tilde{\mathfrak{A}}[\tilde{\mathfrak{U}}] \subset C_{loc}(\mathbb{T}, E)$ for any compact $\Xi \subset E$;*
- 3) *If (fg) holds, then the map $\eta \mapsto x^\eta$ of $\tilde{\mathfrak{U}}$ to $C_{loc}(\mathbb{T}, \mathbb{X})$ and the map $\eta \mapsto \tilde{J}^\eta$ of $\tilde{\mathfrak{U}}$ to $C_{loc}(\mathbb{T}, \mathbb{R})$ are continuous.*

Since the proof of this proposition only plays an auxiliary role, it was repositioned to Appendix. Let us also note that embedding of the initial space \mathfrak{U} of admissible controls into a space with a more convenient topology is a well-known trick; see, for example, [27, 46], and [15, 19, 21], [6, Sect. 8] for infinite horizon problems. A weak compactness was used, for example, in [12, 16, 22, 33].

2.2 Stability and thin tubes of solutions

Let $w : \mathbb{T} \times \mathbb{U} \rightarrow \mathbb{T}$ be an integrally bounded (on each compact subset) Carathéodory map. For all $\tau \in \mathbb{T}$ and $\eta \in \tilde{\mathfrak{U}}$, let us introduce

$$\mathfrak{L}_w[\eta](\tau) \triangleq \int_0^\tau \int_{U(t)} w(t, u) d\eta(t) dt.$$

Let us assume that $\mathfrak{L}_w[\tilde{u}^0] \equiv 0$, and for every $\eta \in \tilde{\mathfrak{U}}$ from $\mathfrak{L}_w[\eta](\tau) = 0$ for all $\tau \in \mathbb{T}$ it follows that η equals \tilde{u}^0 a.e. on $[0, \tau]$. The set of such w is denoted by $(Null)(u^0)$.

For every position $(\vartheta^*, y^*) \in \mathbb{T} \times E$, there exists a unique solution $y \in C(\mathbb{T}, E)$ of the equation

$$\dot{y} = a(t, y(t), u^0(t)), \quad y(\vartheta^*) = y^*. \quad (3c)$$

The solution continuously depends on (ϑ^*, y^*) . Let us denote its initial position $y(0)$ by $\varkappa(\vartheta^*, y^*)$.

Proposition 2 *Let U be a compact-valued map, and its graph is Borel set. Let Ξ be a compact subset of E .*

Then, there exists $w^0 \in (Null)(u^0)$ such that for arbitrary $\eta \in \tilde{\mathfrak{U}}$, $T \in \mathbb{T}$ for every solution y of (3b) from $\varkappa(\vartheta, y(\vartheta)) \in \Xi$ for all $\vartheta \in [0, T]$ it follows that

$$\|\varkappa(\vartheta, y(\vartheta)) - y(0)\|_E \leq \mathfrak{L}_{w^0}[\eta](\vartheta) \quad \forall \vartheta \in [0, T].$$

In the geometric sense, this proposition means that if a solution $y|_{[0, T]}$ from the funnel $\tilde{\mathfrak{A}}[\eta]$ does not escape the area $\tilde{\mathfrak{A}}[u^0]$, then it also does not escape the tube of solutions of (3c), breadth of which (at $t = 0$) does not surpass $\mathfrak{L}_{w^0}[\eta](T)$. See the proof in Appendix.

3 τ -vanishing Lagrange multiplier as a necessary condition

3.1 The core relations of the Maximum Principle

In what follows, we consider the shadow price ψ a covector (a row vector); however, we will still write $x \in \mathbb{X}$, $\psi \in \mathbb{X}$ and will not distinguish between the space \mathbb{X} and its conjugate space in the sense of sets.

Let the Hamilton–Pontryagin function $\mathcal{H} : \mathbb{X} \times \mathbb{T} \times \mathbb{U} \times \mathbb{T} \times \mathbb{X} \rightarrow \mathbb{R}$ be given by

$$\mathcal{H}(x, t, u, \lambda, \psi) \triangleq \psi f(t, x, u) + \lambda g(t, x, u).$$

Let us introduce the relations and boundary condition:

$$\dot{x}(t) = f(t, x(t), u(t)); \quad (4a)$$

$$\dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), t, u(t), \lambda, \psi(t)); \quad (4b)$$

$$\sup_{p \in U(t)} \mathcal{H}(x(t), t, p, \lambda, \psi(t)) = \mathcal{H}(x(t), t, u(t), \lambda, \psi(t)); \quad (4c)$$

$$x(0) = x_{**}, \quad \|\psi(0)\|_{\mathbb{X}} + \lambda = 1. \quad (5a)$$

It is easily seen that, for each $u \in \mathfrak{U}$ for each initial condition, system (4a)–(4b) has a local solution, and each solution of these relations can be extended to the whole \mathbb{T} . Let us denote by \mathfrak{V} the family of all solutions $(x, u, \lambda, \psi) \in C_{loc}(\mathbb{T}, \mathbb{X}) \times \mathfrak{U} \times [0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X})$ of system (4a)–(4b), (5a) on \mathbb{T} . Let us denote by \mathfrak{Z} the set of solutions from \mathfrak{V} such that (4c) also holds a.e. on \mathbb{T} .

Let us embed the sets \mathfrak{V} and \mathfrak{Z} into $C_{loc}(\mathbb{T}, \mathbb{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X})$ by the mapping $(Id, \tilde{\delta}, Id, Id)$; denote closures of their images by $\tilde{\mathfrak{V}}$ and $\tilde{\mathfrak{Z}}$, respectively; then, $\tilde{\mathfrak{V}}$ and $\tilde{\mathfrak{Z}}$ are compacts.

By Proposition 1, for every $(x, \eta, \lambda, \psi) \in \tilde{\mathfrak{V}}$, the following relations hold: (5a), $\widetilde{(4a)}$ – $\widetilde{(4b)}$; for $(x, \eta, \lambda, \psi) \in \tilde{\mathfrak{Z}}$, we also have $\widetilde{(4c)}$, i.e.,

$$\sup_{p \in U(t)} \mathcal{H}(x(t), t, p, \lambda, \psi(t)) = \int_{U(t)} \mathcal{H}(x(t), t, u, \lambda, \psi(t)) d\eta(t). \quad \widetilde{(4c)}$$

Moreover, Proposition 1 implies that all solutions of these equations depend on both controls $u \in \tilde{\mathfrak{U}}$ and initial conditions continuously on any compact.

A nontrivial Lagrange multiplier $(\lambda, \psi) \in [0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X})$ is called a *Lagrange multiplier associated with* (x^0, u^0) if $(x^0, u^0, \lambda, \psi)$ is a solution of the core Maximum Principle, i.e. the system (4a)–(4c). It is convenient to denote by Λ the family of all Lagrange multipliers $(\lambda, \psi) \in \{0, 1\} \times C_{loc}(\mathbb{T}, \mathbb{X})$ associated with (x^0, u^0) such that

$$\lambda = 1 \text{ or } (\lambda = 0 \text{ and } \|\psi(0)\|_{\mathbb{X}} = 1). \quad (5b)$$

For each $\xi \in \mathbb{X}$, let us also define solutions $x_\xi \in C(\mathbb{T}, \mathbb{X})$, $A_\xi \in C(\mathbb{T}, \mathbb{L})$ of the following equations:

$$\dot{x}_\xi(t) = f(t, x_\xi(t), u^0(t)) \quad x_\xi(0) = x_{**} + \xi, \quad (6a)$$

$$\dot{A}_\xi(t) = \frac{\partial f}{\partial x}(t, x_\xi(t), u^0(t)) A_\xi(t) \quad A_\xi(0) = 1_{\mathbb{L}}. \quad (6b)$$

For every $T \in \mathbb{T}$, consider the covector

$$I_\xi(T) \triangleq \int_0^T \frac{\partial g}{\partial x}(t, x_\xi(t), u^0(t)) A_\xi(t) dt.$$

Similarly, for each $u \in \mathfrak{U}$, let us introduce a matrix function A^u and a covector function I^u by the relations

$$\begin{aligned} \dot{A}^u(t) &= \frac{\partial f}{\partial x}(t, x^u(t), u(t)) A^u(t), & A^u(0) &= 1_{\mathbb{L}}, \\ I^u(T) &\triangleq \int_0^T \frac{\partial g}{\partial x}(t, x^u(t), u(t)) A^u(t) dt & \forall T \in \mathbb{T}. \end{aligned} \quad (6c)$$

In addition, we call $x^\eta, A^\eta, \psi^\eta, I^\eta$ the solutions of the corresponding \sim -equations, or, equivalently, the limits, uniform on compacts, of x^u, A^u, ψ^u, I^u as $\tilde{\delta}(u) \rightarrow \eta$ in the $*$ -weak topology of $\tilde{\mathfrak{U}}$.

Expressing the solution of linear equation (4b) through (6c) (or (6b)), then any shadow price ψ has the form

$$\psi(T) = (\psi(0) - \lambda I(T)) A^{-1}(T) \quad \forall T \in \mathbb{T}; \quad (6d)$$

and we can reformulate the result of [28] in the following way:

Theorem 1 *Under conditions (u), (fg), for any τ -optimal pair $(x^0, u^0) \in C_{loc}(\mathbb{T}, \mathbb{X}) \times \mathfrak{U}$ of problem (1a)–(1b), for some $\lambda^0 \in [0, 1], \psi^0 \in C(\mathbb{T}, \mathbb{X})$, the core relations of the Maximum Principle (4a)–(5a) hold for $(x^0, u^0, \lambda^0, \psi^0)$, i.e., $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$.*

Moreover, up to a positive factor, for some $I_* \in \mathbb{X}, \iota_* \in \mathbb{X}$, one of the two following relations also holds:

$$\lambda^0 = 1, \quad \psi^0(T) = (I_* - I_0(T)) A_0^{-1}(T) \quad \forall T \in \mathbb{T}; \quad (7a)$$

$$\lambda^0 = 0, \quad \psi^0(T) = \iota_* A_0^{-1}(T) \quad \forall T \in \mathbb{T}. \quad (7b)$$

The core relations of the Maximum Principle are incomplete, since (4a)–(5a) do not contain a condition on the right endpoint, or, which is actually equivalent, on I_* or ι_* . The remaining part of the paper is mainly devoted to finding the additional relations at I_* and ι_* with the aid of τ -vanishing Lagrange multiplier.

3.2 Existence of τ -vanishing multipliers

System (4a)–(4b) can be rewritten for $u = u^0$ in the form

$$\dot{\psi}(t) = -\frac{\partial \mathcal{H}}{\partial x}(x(t), t, u^0(t), \lambda, \psi(t)), \quad (8a)$$

$$\dot{x}(t) = f(t, x(t), u^0(t)), \quad (8b)$$

$$\dot{\lambda} = 0. \quad (8c)$$

Definition 2 A nontrivial Lagrange multiplier (λ^0, ψ^0) associated with (x^0, u^0) is called τ -vanishing if (ψ^0, x^0, λ^0) is a pointwise limit of a sequence of solutions $(\psi_n, x_n, \lambda_n)_{n \in \mathbb{N}}$ of system (8a)–(8c) such that $\psi_n(\tau'_n) = 0$ for every $n \in \mathbb{N}$, here $\tau' \subset \tau$. In this case, the shadow price ψ^0 is called τ -vanishing as well.

Geometrically, this property means that the tube of solutions of system (8a)–(8c), however thin (at the initial time), intersects with the hyperplane $\psi = 0_{\mathbb{X}}$ at arbitrarily far time τ_n .

We claim that the existence of τ -vanishing multipliers is a necessary optimality condition. The main work horse of this proof is the following asymptotic condition of optimality structurally similar to [6, Theorem 9.1], [8, Theorem 3].

Proposition 3 Under conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$, for each weight $w \in (Null)(u^0)$, there exist a sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbb{N}} \in \tilde{\mathfrak{Y}}^{\mathbb{N}}$ and a subsequence τ' of τ such that

- 1) for some $(x^0, \tilde{u}^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ it is $(x^n, \eta^n, \lambda^n, \psi^n) \rightarrow (x^0, \tilde{u}^0, \lambda^0, \psi^0)$ in the topology of $C_{loc}(\mathbb{T}, \mathbb{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X})$;
- 2) $\|\mathfrak{L}_w(\eta^n)\|_C \rightarrow 0$;
- 3) $\tilde{\eta}^n(\tau'_n) - J^{u^0}(\tau'_n) \rightarrow 0+$; $\psi^n(\tau'_n) = 0$ for all $n \in \mathbb{N}$.

The proof of this proposition was repositioned to Appendix.

Note that from $\psi^n(0) = -\psi^n(\tau'_n)A^{\eta^n}(\tau'_n) + \psi^n(0)A^{\eta^n}(0) \stackrel{(6d)}{=} \lambda^n I^{\eta^n}(\tau'_n)$, we have $\lambda^n I^{\eta^n}(\tau'_n) \rightarrow \psi^0(0)$.

Let $E = \mathbb{X} \times \mathbb{X} \times \mathbb{T}$, $\Xi \triangleq 2\mathbb{D} \times (x_{**} + 2\mathbb{D}) \times [0, 1]$. To system (4b), (4a), (8c), let us assign the weight w by means of Propositions 2. Substituting this weight into Proposition 3, we obtain

Remark 1 Under conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$ there exist a sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbb{N}} \in \tilde{\mathfrak{Y}}^{\mathbb{N}}$ and a subsequence τ' of τ such that

- 1) for some $(x^0, \tilde{u}^0, \lambda^0, \psi^0) \in \mathfrak{Z}$, it is $(x^n, \eta^n, \lambda^n, \psi^n) \rightarrow (x^0, \tilde{u}^0, \lambda^0, \psi^0)$ in the topology of $C_{loc}(\mathbb{T}, \mathbb{X}) \times \tilde{\mathfrak{U}} \times [0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X})$;
- 2) the graphs of functions (ψ^n, x^n, λ^n) are contained within the thinning funnels of solutions of system (8a)–(8c); i.e., for some sequence $(\delta_n)_{n \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, $\delta_n \downarrow 0$, we have

$$\kappa(t, (\psi^n, x^n, \lambda^n)(t)) \in (\psi^0(0), x_{**}, \lambda^0) + \delta_n \mathbb{D} \times \delta_n \mathbb{D} \times [-\delta_n, \delta_n] \quad \forall t \in \mathbb{T}, n \in \mathbb{N};$$

- 3) $\tilde{\eta}^n(\tau'_n) - J^{u^0}(\tau'_n) \rightarrow 0+$;
- 4) $\lambda^n I^{\eta^n}(\tau'_n) \rightarrow \psi^0(0)$; $\psi^n(\tau'_n) = 0$ for all $n \in \mathbb{N}$.

Note that (λ^0, ψ^0) is nontrivial because it satisfies boundary condition (5a) as well as the multipliers (λ^n, ψ^n) . For every $n \in \mathbb{N}$, consider a solution (ψ_n, x_n, λ^n) of (8a)–(8c) with the initial conditions $(\psi_n(0), x_n(0), \lambda_n) \triangleq \kappa(\tau'_n, (\psi^n(\tau'_n), x^n(\tau'_n), \lambda^n))$. Then $\psi_n(\tau'_n) = 0_{\mathbb{X}}$. Since $(\psi_n(0), x_n(0), \lambda_n) \in (\psi^0(0), x^0(0), \lambda^0) + \delta_n \mathbb{D} \times \delta_n \mathbb{D} \times [-\delta_n, \delta_n]$, i.e., $(\psi_n(0), x_n(0), \lambda_n) \rightarrow (\psi^0(0), x^0(0), \lambda^0)$, and because of the continuous dependency of solutions of (8a)–(8c), (λ^0, ψ^0) is a τ -vanishing Lagrange multiplier.

Theorem 2 Assume that conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$ hold.

Then, there exists a τ -vanishing Lagrange multiplier $(\lambda^0, \psi^0) \in \Lambda$, for example, constructed with a limit of sequences from Remark 1.

Moreover, for every τ -vanishing Lagrange multiplier $(\lambda^0, \psi^0) \in \Lambda$, there exist a subsequence τ' of τ , a converging to $0_{\mathbb{X}}$ sequence $(\xi^n)_{n \in \mathbb{N}} \in \mathbb{X}^{\mathbb{N}}$, a converging to λ^0 sequence $(\lambda^n)_{n \in \mathbb{N}} \in (0, 1]^{\mathbb{N}}$ such that

$$\psi^0(0) = \lim_{n \rightarrow \infty} \lambda^n I_{\xi^n}(\tau'_n); \quad (9a)$$

$$\psi^0(T) = \lim_{n \rightarrow \infty} \lambda^n (I_{\xi^n}(\tau'_n) - I_{\xi^n}(T)) A_{\xi^n}^{-1}(T) \quad (9b)$$

$$= \lim_{n \rightarrow \infty} \lambda^n \int_T^{\tau'_n} \frac{\partial g}{\partial x}(t, x_{\xi^n}(t), u^0(t)) A_{\xi^n}(t) dt A_0^{-1}(T). \quad (9c)$$

and all the limits are uniform on every compact.

If, in addition to that, $\lambda^0 > 0$, then we can assume $\lambda_n = \lambda^0 = 1$.

Proof. The existence of a τ -vanishing Lagrange multiplier (λ^0, ψ^0) is shown above. By multiplying this nontrivial (λ^0, ψ^0) by a certain scalar, we can always provide condition (5a); thus, $(\lambda^0, \psi^0) \in \Lambda$.

Let (λ^0, ψ^0) be a τ -vanishing Lagrange multiplier. The sequences $\tau', (\lambda_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}}$ exist by the definition of a τ -vanishing Lagrange multiplier if we define $\xi^n \triangleq x_n(0) - x^0(0)$ for every $n \in \mathbb{N}$. Since for all $n \in \mathbb{N}$ $\psi_n(\tau'_n) = 0_{\mathbb{X}}$, the Cauchy formula (6d) implies $\psi_n(T) = \lambda^n(I_{\xi^n}(\tau'_n) - I_{\xi^n}(T))A_{\xi^n}^{-1}(T)$ for every $T \in \mathbb{T}$, we have $\psi_n(0) = \lambda^n(I_{\xi^n}(\tau'_n) - I_{\xi^n}(T)) = \lambda^n I_{\xi^n}(\tau'_n)$. Now, uniformity of the limit ψ^0 of ψ_n yields (9a). Substituting this into (6d) we obtain (9c) for every $T \in \mathbb{T}$. What remains follows from the theorem of continuous dependence of solutions on initial conditions, applied to (8a)–(8c) and (6b). \square

3.3 On different topologies for the set of generalized controls

Consider a weight $w^0 \in (Null)(u^0)$. Define w^1 by the rule $w^1(t, u) \triangleq w^0(t, u) + \|u - u^0(t)\|$ for every $(t, u) \in \mathbb{T} \times \mathbb{U}$. Then, for a subsequence $(u_n)_{n \in \mathbb{N}} \in \mathfrak{U}^{\mathbb{N}}$, from $\|\mathfrak{L}_{w^1}[\tilde{\delta} \circ u_n]\|_C \rightarrow 0$ it follows that $\|u_n - u^0\|_{\mathcal{L}^1(\mathbb{T}, \mathbb{U})} \rightarrow 0$ (certainly, this does not imply that $u^0 \in \mathcal{L}^1(\mathbb{T}, \mathbb{U})$). Similarly, for any $p \in (0, \infty)$, $\nu \in B_{loc}(\mathbb{T}, \mathbb{R}_{>0})$, replacing $\|u - u^0(t)\|$ with $\nu(t)\|u - u^0(t)\|^p$ guarantees the convergence of $u_n - u^0 \rightarrow 0$ in the topology of $\mathcal{L}^p_{\nu}(\mathbb{T}, \mathbb{U})$. For every interval $\mathfrak{T} \subset \mathbb{T}$, this extended metric also induces the extended distance $\varrho(\eta, u^0; \tilde{\mathcal{L}}^p_{\nu}(\mathfrak{T}, \mathbb{U}))$ on $\tilde{\mathfrak{U}}$ by the rule

$$\varrho(\eta, u^0; \tilde{\mathcal{L}}^p_{\nu}(\mathfrak{T}, \mathbb{U})) \triangleq \left(\int_{\mathfrak{T}} \nu(t) \int_{U(t)} \|u - u^0(t)\|^p d\eta(t) dt \right)^{1/p} \quad \forall \eta \in \tilde{\mathfrak{U}}.$$

Addition of the summand $\nu(t)R^p(t, u)$ (see (32b)) provides the uniform convergence $\|\dot{y}(t) - a(t, y(t), u^0(t))\|_{\mathcal{L}^p_{\nu}(\mathbb{T}, \mathbb{X})} \rightarrow 0$ by all $\eta \in \tilde{\mathfrak{U}}, y \in \mathfrak{A}[\eta]$ such that $y(0) \in \Xi$.

Let us replace the weight w from Proposition 3 and Remark 1 by stronger ones if necessary. Then there exists a τ -vanishing Lagrange multiplier (λ^0, ψ^0) as the limit of sequences from Remark 1. Now we have

Remark 2 Assume that conditions **(u)**, **(fg)**, **(τ)** hold. Then there exists a τ -vanishing multiplier (λ^0, ψ^0) associated with (x^0, u^0) such that for this multiplier, the conclusion of Remark 1 holds and, moreover, the following convergences are guaranteed: $\varrho(\eta^n, u^0; \tilde{\mathcal{L}}^p_{\nu}(\mathbb{T}, \mathbb{U})) \rightarrow 0$, $\|\dot{x}^n(t) - f(t, x^n(t), u^0(t))\|_{\mathcal{L}^p_{\nu}(\mathbb{T}, \mathbb{X})} \rightarrow 0_{\mathbb{X}}$.

The condition **(u)** implies that, a.a. $t \in \mathbb{T}$, the controls are chosen from the compact $U(t)$. Let us weaken this assumption to the following:

Condition (u_{σ}) : U is a closed-valued map, and its graph is Borel set.

We shall still assume the conditions **(fg)**, **(τ)** to hold. A nondecreasing sequence $(U^{(r)})_{r \in \mathbb{N}}$ of locally bounded compact-valued maps is given by

$$U^{(r)}(t) \triangleq \{u \in U(t) \mid \|u - u^0(t)\| < r\} \quad \forall t \in \mathbb{N}, r \in \mathbb{N}.$$

Let the set $\mathfrak{U}^{(r)}$ be the set of all Borel measurable selectors of the multi-valued map $U^{(r)}$. Then for all $r \in \mathbb{N}$ $u^0 \in \mathfrak{U}^{(r)} \subset \mathfrak{U}^{(r+1)}$ and $U \equiv \cup_{r \in \mathbb{N}} U^{(r)}$ hold; now, we have $\mathfrak{U}^{(\infty)} \triangleq \cup_{r \in \mathbb{N}} \mathfrak{U}^{(r)} \equiv \mathfrak{U}$.

Repeating the reasonings of Sect 2, for every $r \in \mathbb{N} \cup \{\infty\}$, we can construct sets $\tilde{\mathfrak{U}}^{(r)}$ and images $\mathfrak{U}_n^{(r)} \triangleq \pi_n(\mathfrak{U}^{(r)}), \tilde{\mathfrak{U}}_n^{(r)} \triangleq \tilde{\pi}_n(\tilde{\mathfrak{U}}^{(r)})$. Denote by $\tilde{\mathfrak{U}}$ the set of all maps η from \mathbb{T} into the set of Radon probability measures

over \mathbb{U} such that $\eta|_{[0,n]} \in \widetilde{\mathfrak{U}}_n^{(\infty)}$ for every $n \in \mathbb{N}$. The topology of this set is the weakest topology in which $\mathfrak{U}^{(r)}$ could be continuously embedded into $\widetilde{\mathfrak{U}}$. Note that under our definition, $\widetilde{u}^0 \in \widetilde{\delta}(\mathfrak{U}^{(r)}) \subset \widetilde{\delta}(\mathfrak{U})$ for all $r \in \mathbb{N}$.

To system (4b),(4a),(8c), let us assign the weight w by means of Propositions 2. Note that this weight is independent of r . For the sequence τ , for each $\widetilde{\mathfrak{U}}^{(r)}$, we have Remark 1; in particular, there exist a time $t_r \in \tau$, $t_r > r$, a τ -vanishing Lagrange multiplier (λ^r, ψ^r) , and a solution $(x^r, \eta^r, \widetilde{\psi}^r, \widetilde{\lambda}^r) \in \widetilde{\mathfrak{V}}$ with the properties

$$\sup_{p \in U^{(r)}(t)} \mathcal{H}(x(t), t, p, \lambda^r, \psi^r(t)) = \mathcal{H}(x(t), t, u^0(t), \lambda^r, \psi^r(t)) \forall a.a. t \in \mathbb{T} \quad (10a)$$

$$\|\mathfrak{L}_w(\eta^r)\|_C < 1/r, \|\widetilde{x}^r - x^r\|_{C([0,r],\mathbb{X})} < 1/r, \|\widetilde{\psi}^r - \psi^r\|_{C([0,r],\mathbb{X})} < 1/r, \quad (10b)$$

$$\|\mathfrak{x}(t_r, (\widetilde{\psi}^r(t_r), \widetilde{x}^r(t_r), \widetilde{\lambda}^r)) - (\psi^0, x^{**}, \lambda^0)\|_E < 1/r, \quad (10c)$$

$$0 \leq \widetilde{J}^{\eta^r}(t_r) - J^{u^0}(t_r) < 1/r, \quad \widetilde{\psi}^r(t_r) = 0_{\mathbb{X}}. \quad (10d)$$

Passing to the limit, we obtain $\eta^r \rightarrow \widetilde{u}^0$ from $\|\mathfrak{L}_w(\eta^r)\|_C < 1/r$. Passing to the subsequence $\tau' \subset (t_r)_{r \in \mathbb{N}} \subset \tau$, we can provide the monotonicity of t_r and convergence of the sequence $(\lambda^r, \psi^r)_{r \in \mathbb{N}} \in ((0, 1] \times C_{loc}(\mathbb{T}, \mathbb{X}))^{\mathbb{N}}$ to certain (λ^0, ψ^0) . Under these assumptions, we immediately see that (ψ^0, x^0, λ^0) is the solution of (8a)-(8c) that satisfies (5a). Now the sequence $(\widetilde{x}^r, \eta^r, \widetilde{\psi}^r, \widetilde{\lambda}^r)_{r \in \mathbb{N}}$ converges, by (10b), to $(x^0, \widetilde{u}^0, \lambda^0, \psi^0)$. Passing to the pointwise limit in (10a), we obtain for $(x^0, \widetilde{u}^0, \lambda^0, \psi^0)$ the property (4c). Thus we proved items 1) and 2) of Remark 1. Since the limit of $(\widetilde{x}^r, \eta^r, \widetilde{\psi}^r, \widetilde{\lambda}^r)_{r \in \mathbb{N}}$ and $(x^0, \eta^r, \psi^r, \lambda^r)_{r \in \mathbb{N}}$ is the same, items 3) and 4) follow from (10c) and (10d) respectively.

Consider again the solutions (ψ_n, x_n, λ^n) of (8a)-(8c) for the initial conditions $(\psi_n(0), x_n(0), \lambda_n) \triangleq \mathfrak{x}(\tau'_n, (\widetilde{\psi}^n(\tau'_n), \widetilde{x}^n(\tau'_n), \widetilde{\lambda}^n))$. Then (λ^0, ψ^0) is a τ -vanishing Lagrange multiplier and Theorem 2 holds under condition (\mathbf{u}_σ) . Thus,

Corollary 1 *Condition (\mathbf{u}) in Remark 1, Theorem 2 could be replaced with (\mathbf{u}_σ) .*

Corollary 2 *Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg})$ hold. Let a pair $(x^0, u^0) \in C_{loc}(\mathbb{T}, \mathbb{X}) \times \mathfrak{U}$ be weakly uniformly overtaking optimal for problem (1a)-(1b).*

Then, for some unbounded sequence $\tau = (\tau_n)_{n \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, there exists a τ -vanishing Lagrange multiplier $(\lambda^0, \psi^0) \in \Lambda$.

Corollary 3 *Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg})$ hold. Let a pair $(x^0, u^0) \in C_{loc}(\mathbb{T}, \mathbb{X}) \times \mathfrak{U}$ be uniformly overtaking optimal for problem (1a)-(1b).*

Then, for each unbounded sequence $\tau = (\tau_n)_{n \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, there exists a τ -vanishing Lagrange multiplier $(\lambda^0, \psi^0) \in \Lambda$.

4 Properties of τ -vanishing Lagrange multipliers

4.1 On stable shadow prices

Consider the boundary conditions

$$\lim_{t \rightarrow \infty} \psi(t) = 0, \quad (11a)$$

$$\liminf_{n \rightarrow \infty} \|\psi^0(\tau_n)\|_{\mathbb{X}} = 0. \quad (11b)$$

Definition 3 The component ψ^0 of a solution $y^0 = (\psi^0, x^0, \lambda^0)$ of system (8a)–(8c) is said to be *Lyapunov stable in domain Ξ* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for each solution $y = (\psi, x, \lambda)$ of system (8a)–(8c) from $\|y(0) - y^0(0)\|_E < \delta$, $y(0) \in \Xi$ it follows that $\|\psi^0(s) - \psi(s)\|_{\mathbb{X}} < \varepsilon$ for all $s \in \mathbb{T}$.

Corollary 4 Assume that conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. Let for some solution (ψ, x^0, λ) of system (8a)–(8c) the component ψ be Lyapunov stable in the domain $\mathbb{X} \times \mathbb{X} \times [0, 1]$.

Then all τ -vanishing multipliers $(\lambda^0, \psi^0) \in \Lambda$ satisfy the condition (11b).

Proof. Since equation (8a) is linear, the Lyapunov stability of ψ for some solution (ψ, x^0, λ) of system (8a)–(8c) yields the Lyapunov stability of this component for all solutions of system (8a)–(8c).

Consider every τ -vanishing multiplier (λ^0, ψ^0) and the sequences $\tau', (\lambda_n)_{n \in \mathbb{N}}, (x_n)_{n \in \mathbb{N}}, (\psi_n)_{n \in \mathbb{N}}$ from its definition. Then, $y_n = (\psi_n(0), x_n(0), \lambda_n) \rightarrow y^0 = (\psi^0(0), x^0(0), \lambda^0)$, and by definition of Lyapunov stability for some $N \in \mathbb{N}$ for all $n \in \mathbb{N}, n > N$ $\|\psi^0(\tau'_n)\|_{\mathbb{X}} = \|\psi^n(\tau'_n) - \psi^0(\tau'_n)\|_{\mathbb{X}} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we have shown (11b) for all τ -vanishing multipliers. \square

Note that since (4b) is linear, the partial stability of the variable ψ implies its boundedness. Therefore, the proved proposition is useless if all shadow prices are unbounded. Note that, as follows from [43, Example 5.1], for a weakly uniformly overtaking optimal control u^0 , a $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$ that satisfies (11b) may not satisfy stronger condition (11a).

The stability condition can be replaced with a condition which is stronger but much easier to check.

Corollary 5 Assume that conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. If the functions L_K^f, L_K^g are independent of the compact K , and these functions are summable on \mathbb{T} (see [37, Hypotesis 3.1 (iv)]), then any τ -vanishing multiplier satisfies condition (11a).

Proof. Let (ψ^0, λ^0) be a τ -vanishing multiplier. Let $\xi_0 \triangleq (\psi^0(0), x^0(0), \lambda^0)$, $\Xi \triangleq \xi_0 + \mathbb{D} \times \mathbb{D} \times [-1, 1]$. By [37, (3.3)] there exists a summable function $\omega \in \mathfrak{L}^1(\mathbb{T}, \mathbb{T})$ such that $\dot{\psi}(t) \leq \omega(t)$ for a.a. $t \in \mathbb{T}$ for all solution (ψ, x, λ) of system (8a)–(8c) if $\xi \triangleq (\psi(0), x(0), \lambda) \in \Xi$. Now for each pair $(t_1, t_2) \in \mathbb{T}, t_1 \leq t_2$,

$$\|\psi(t_1) - \psi^0(t_2)\|_{\mathbb{X}} \leq \|\psi - \psi^0\|_{C([0, t_1], \mathbb{X})} + 2 \int_{t_1}^{\infty} \omega(t) dt$$

if $\xi \in \Xi$. For each $\varepsilon > 0$ there exists $T \in \mathbb{T}$ such that the second summand does not exceed $\varepsilon/2$ if $t_1 > T$; now there exists $r \in \mathbb{T}$ such that $\|\psi - \psi^0\|_{C([0, T], \mathbb{X})}$ does not exceed $\varepsilon/2$ if $\|\xi - \xi^0\|_E < r$. Then, setting $t_1 = t_2$, we obtain $\|\psi - \psi^0\|_C \leq \varepsilon$ if $\|\xi_1 - \xi_2\|_E < r$, i.e., the component ψ^0 is Lyapunov stable. By Corollary 4, (11b) holds, and $\|\psi^0(t_1)\| < \varepsilon/2$ for some $t_1 \in \mathbb{T}, t_1 > T$.

Then, setting $\xi = (\psi^0(0), x^0(0), \lambda^0)$, we obtain $\|\psi^0(t_2)\|_{\mathbb{X}} = \|\psi^0(t_2) - \psi^0(t_1)\|_{\mathbb{X}} + \|\psi^0(t_1)\|_{\mathbb{X}} < \varepsilon$ if $t_2 > t_1$.

Thus (11a) holds. \square

The even more strong conditions used for proving transversality condition (11a) can be seen, for example, in [50, (A3)] (the Lipschitz constants L_K^g, L_K^f are required to decrease exponentially with time).

4.2 Degenerate τ -vanishing Lagrange multipliers

Remark 3 Assume that conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. If for some $\tau' \subset \tau$

$$\limsup_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \|I_{\xi}(\tau'_n)\|_{\mathbb{X}} < \infty \quad (12)$$

then the pair (x^0, u^0) is normal, there exists a τ -vanishing multiplier $(1, \psi^0) \in \Lambda$.

Moreover, if $\limsup_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \|I_{\xi}(\tau_n)\|_{\mathbb{X}} < \infty$, then every τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$ satisfies $\lambda^0 = 1$.

On the other hand, if $\lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \|I_{\xi}(\tau_n)\|_{\mathbb{X}} = \infty$, then every τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$ satisfies $\lambda^0 = 0$.

Proof. By Theorem 2 there exists a τ -vanish multiplier $(\lambda^0, \psi^0) \in \Lambda$ satisfying (9a), but for each such multiplier we have $\lambda^n \|I_{\xi_n}(\tau'_n)\|_{\mathbb{X}} = \|\psi^n(0)\|_{\mathbb{X}} \rightarrow \|\psi^0(0)\|_{\mathbb{X}}$; then $\lambda^0 = 0$ iff $(\|I_{\xi_n}(\tau'_n)\|_{\mathbb{X}})_{n \in \mathbb{N}} \uparrow \infty$. \square

There are many conditions that provide nondegeneracy of the problem; in connection with this, note papers [6, 8, 10, 11, 38]. The connection between the normality of the problem and finiteness of I_0 seems to be noted for the first time in [6, (3.24)]. Condition (12) develops this approach, actually demanding I_{ξ} to be locally bounded. As we are going to show below, many sufficient conditions of nondegeneracy for the optimal problem can be obtained from (12). However, there are other ways to prove the nondegeneracy. For example, [6, Theorem 5.1] uses the smoothness of the objective value function, and [6, Theorem 10.1] and [8, Theorem 5] use the monotonicity of the functions f and g in x and the stationarity condition.

Note that although the examples of abnormal problems are well known ([28, 6, 35]), additional relations of the Maximum Principle for such problems did not receive much attention from researchers; the author only knows of the dual problem construction in paper [35]. Let us apply Theorem 2 to these problems.

Consider a degenerate τ -vanishing solution $(x^0, u^0, 0, \psi^0) \in \mathfrak{Z}$. Then, from (5a) we have $\psi^0(0) = 1$, and Theorem 2 yields

$$\psi^0(0) \stackrel{(9a)}{=} \frac{\psi^0(0)}{\|\psi^0(0)\|_{\mathbb{X}}} = \lim_{n \rightarrow \infty} \frac{\lambda_n I_{x_n(0)}(\tau'_n)}{\|\lambda_n I_{x_n(0)}(\tau'_n)\|_{\mathbb{X}}} = \lim_{n \rightarrow \infty} \frac{I_{x_n(0)}(\tau'_n)}{\|I_{x_n(0)}(\tau'_n)\|_{\mathbb{X}}} \quad (13)$$

provided $x_n(0) \rightarrow x^0(0)$. Using Remark 3, we finally obtain

Corollary 6 *Let $(\mathbf{u}_{\sigma}), (\mathbf{f}g)$, (τ) hold. Let*

$$\lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \|I_{\xi}(\tau_n)\|_{\mathbb{X}} = \infty, \quad \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \frac{I_{\xi}(\tau_n)}{\|I_{\xi}(\tau_n)\|_{\mathbb{X}}} = \iota_*$$

for some vector $\iota_ \in \mathbb{X}$.*

Then, there is unique τ -vanishing multiplier $(0, \psi^0) \in \Lambda$, and ι_ and ψ^0 are connected by (7b).*

4.3 Monotonic case

Consider a nonempty convex closed cone \mathfrak{C} . The cone orderings \succcurlyeq, \succ of \mathbb{X} induced by \mathfrak{C} are the relations defined as follows: for all $x, y \in \mathbb{X}$,

$$(x \succcurlyeq_{\mathfrak{C}} y) \Leftrightarrow (x - y \in \mathfrak{C}), \quad (x \succ_{\mathfrak{C}} y) \Leftrightarrow (x - y \in \text{int } \mathfrak{C}).$$

The pre-orders on \mathbb{L} defined as follows: for $B, C \in \mathbb{L}$,

$$(B \succcurlyeq_{\mathfrak{C}} C) \Leftrightarrow ((B - C)x \in \mathfrak{C} \quad \forall x \in \mathfrak{C}), \\ (B \succ_{\mathfrak{C}} C) \Leftrightarrow ((B - C)x \in \text{int } \mathfrak{C} \quad \forall x \in \text{int } \mathfrak{C}).$$

Note that $1_{\mathbb{L}} \succ_{\mathfrak{C}} 0_{\mathbb{L}}$, $1_{\mathbb{L}} \succ_{\mathfrak{C}} 0_{\mathbb{L}}$.

The conjugate cone of \mathfrak{C} is defined by $\mathfrak{C}^{\perp} \triangleq \{x \in \mathbb{X} \mid \forall y \in \mathfrak{C} \, xy \geq 0\}$.

Proposition 4 *Assume that conditions (\mathbf{u}_{σ}) , (\mathbf{fg}) , (τ) hold. Assume that there exists Carat  odory function $d : \mathbb{T} \times \mathbb{X} \rightarrow \mathbb{R}$ such that for all $x \in \mathbb{X}$ and a.a. $t \in \mathbb{T}$ the following relation holds:*

$$\frac{\partial g}{\partial x}(t, x, u^0(t)) \succ_{\mathfrak{C}^{\perp}} 0_{\mathbb{L}}, \quad \frac{\partial f}{\partial x}(t, x, u^0(t)) \succ_{\mathfrak{C}} d(t, x)1_{\mathbb{L}}.$$

Then, there exists a τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$, and for any such multiplier, we have $\psi^0 \succ_{\mathfrak{C}^{\perp}} 0_{\mathbb{X}}$, and $\psi^0(0) \in \mathfrak{C}^{\perp}$.

Moreover, if $\lambda^0 > 0$ (for example, if (12) holds), then for all $y \in \mathfrak{C}$

$$\limsup_{t \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} I_{\xi}(t)y \geq \psi^0(0)y \geq \lim_{t \rightarrow \infty} I_0(t)y \geq 0, \quad (14)$$

and all limits in (14) are correctly defined.

If, in addition, there exists a Lebesgue point $t^* \in \mathbb{T}$ for the function u^0 such that

$$\frac{\partial g}{\partial x}(t^*, x^0(t^*), u^0(t^*)) \succ_{\mathfrak{C}^{\perp}} 0_{\mathbb{L}},$$

then $\psi^0|_{[0, t^*]} \succ_{\mathfrak{C}^{\perp}} 0_{\mathbb{X}}$; in particular, $\psi^0(0) \in \text{int } \mathfrak{C}^{\perp}$. If such Lebesgue point t^* exists on every infinite interval, then $\psi^0 \succ_{\mathfrak{C}^{\perp}} 0$.

Proof. Fix arbitrary $\xi \in \mathbb{X}$, $T > 0$, $\vartheta > T$. Denote by $F_{\xi}(t)$ the matrix $\frac{\partial f}{\partial x}(t, x_{\xi}(t), u^0(t))$, and by m_{ξ} the measurable function $t \mapsto -d(t, x_{\xi}(t))$; by condition, $F_{\xi} + m_{\xi}(t)1_{\mathbb{L}} \succ_{\mathfrak{C}} 0_{\mathbb{L}}$. Now, let us consider a solution $P(t)$ of the equation

$$\dot{P} = (F_{\xi}(t) + m_{\xi}(t)1_{\mathbb{L}})P, \quad P(T) = 1_{\mathbb{L}}, \quad t \geq T;$$

then $P(t) \succ_{\mathfrak{C}} 1_{\mathbb{L}}$ for all $t \in (T, \vartheta]$. But the solution P is the product of two nonnegative solutions of the equations $\dot{Q} = F_{\xi}(t)Q$, $Q(T) = 1_{\mathbb{L}}$, and $\dot{r}_{\xi} = m_{\xi}(t)r_{\xi}$, $r_{\xi}(T) = 1$. Thus, $P(\vartheta) = r_{\xi}(\vartheta)Q(\vartheta) = r_{\xi}(\vartheta)A_{\xi}(\vartheta)A_{\xi}^{-1}(T)$, and $P(\vartheta) \succ_{\mathfrak{C}} 1_{\mathbb{L}}$ implies $A_{\xi}(\vartheta)A_{\xi}^{-1}(T) = Q(\vartheta) = P(\vartheta)/r_{\xi}(\vartheta) \succ_{\mathfrak{C}} 1_{\mathbb{L}}/r_{\xi}(\vartheta)$ for all $\vartheta > T$. In particular, for all $y \in \mathfrak{C}$, we have $A_{\xi}(\vartheta)A_{\xi}^{-1}(T)y \succ_{\mathfrak{C}} y/r_{\xi}(\vartheta)$, whence

$$\frac{dI_{\xi}(t)}{dt}A_{\xi}^{-1}(T)y = \frac{\partial g}{\partial x}(t, x_{\xi}(t), u^0(t))A_{\xi}(t)A_{\xi}^{-1}(T)y \geq \frac{\partial g}{\partial x}(t, x_{\xi}(t), u^0(t))\frac{y}{r_{\xi}(t)} \geq 0 \quad (15)$$

for all $\xi \in \mathbb{X}$, $y \in \mathfrak{C}$, $T \in \mathbb{T}$, $t > T$; thus, for $T = 0$, we have $\frac{dI_{\xi}(t)}{dt} \in \mathfrak{C}^{\perp}$, hence the functions $I_{\xi}y$, $I_{\xi}A_{\xi}^{-1}(T)y$ are monotonic for all $\xi \in \mathbb{X}$, $T \in \mathbb{T}$, $y \in \mathfrak{C}$.

By Theorem 2, there exists a τ -vanishing multiplier $(\psi^0, \lambda^0) \in \Lambda$. Moreover, each such multiplier $(\psi^0, \lambda^0) \in \Lambda$ satisfies formula (9c) for certain sequences λ^n and ξ_n . However, the integrand of (9c) lies in \mathfrak{C}^{\perp} . Passing to the limit as $n \rightarrow \infty$, we obtain $\psi^0 \succ_{\mathfrak{C}^{\perp}} 0_{\mathbb{X}}$.

Fix the basis of $\text{span } \mathfrak{C}$ made of the vectors $y \in \mathfrak{C}$; now, for every such vector y , the functions $I_{\xi}y$ are monotonic, and

$$\limsup_{t \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \lambda_0 I_{\xi}(t)y \geq \lim_{n \rightarrow \infty} \lambda_0 I_{\xi_n}(\tau'_n)y \stackrel{(9a)}{=} \psi^0(0)y;$$

we obtain the first estimate from (14).

Fix any $T \in \mathbb{T}$, $y \in \mathfrak{C}$. Now, monotonicity of $I_\xi A_\xi^{-1}(T)y$ yields

$$\begin{aligned} \psi^0(T)y &\stackrel{(9b)}{=} \lim_{n \rightarrow \infty} \lambda^n (I_{\xi_n}(\tau'_n) - I_{\xi_n}(T)) A_{\xi_n}^{-1}(T)y \geq \lim_{n \rightarrow \infty} \lambda^n (I_{\xi_n}(t) - I_{\xi_n}(T)) A_{\xi_n}^{-1}(T)y \\ &= \lambda^0 (I_0(t) - I_0(T)) A_0^{-1}(T)y \stackrel{(15)}{\geq} \lambda^0 \int_T^t \frac{\partial g}{\partial x}(\vartheta, x^0(\vartheta), u^0(\vartheta)) \frac{y d\vartheta}{r_0(\vartheta)} \geq 0 \quad \forall t > T. \end{aligned} \quad (16)$$

Substituting $T = 0$ and passing to the limit as $t \rightarrow \infty$, we obtain the lower estimate from (14).

If $\lambda^0 > 0$, and, in addition, there exists the Lebesgue point t^* with the required property, then for all points $T \leq t^*$, $t > t^*$, sufficiently close to t^* , integration on $[T, t]$ yields " $>$ " instead of " \geq " in the latter inequality of (16). Since by (15) this integrand is nonnegative for all $t \in \mathbb{T}$, the same is true for all $T \leq t^*$, $t > t^*$, whence we obtain $\psi^0|_{[0, t^*]} \succ_{\mathfrak{C}} 0_{\mathbb{X}}$.

Regarding the latter point, note that if we have $\psi(t) \not\succ_{\mathfrak{C}} 0_{\mathbb{X}}$ for some $t \in \mathbb{T}$, then taking t^* from (t, ∞) yields a contradiction. \square

Remark 4 For $\psi^0(0) \succ_{\mathfrak{C}} 0$, it is sufficient to find for each vector y_i from some basis of $\text{span } \mathfrak{C}$ Lebesgue point t_i^* with the property $\frac{\partial g}{\partial x}(t_i^*, x^0(t_i^*), u^0(t_i^*)) y_i > 0$.

Let the right-hand side of the dynamics equation and the integrand of the objective functional be monotonic in x . This case frequently arises in economical applications, and monotonicity simplifies its analysis. It seems that the first to note the peculiarities of this case and to investigate it were Aseev, Kryazhinskii, and Taras'ev in their paper [9]. These were followed by papers [4, 47]; the most general cases were considered in [6, 8].

Fix the cone $\mathfrak{C} \triangleq \mathbb{T}^{\dim \mathbb{X}}$. In this case, $\mathfrak{C}^\perp = \mathfrak{C}$. Replace $\succ_{\mathbb{T}^{\dim \mathbb{X}}}$, $\succ_{\mathbb{T}^{\dim \mathbb{X}}}$ with \succ, \succ . We obtain

Corollary 7 Assume conditions $(\mathbf{u}_\sigma), (\mathbf{f}\mathbf{g}), (\tau)$ hold. Assume that, for all $x \in \mathbb{X}$ and for a.a. $t \in \mathbb{T}$, the matrix $\frac{\partial f}{\partial x}(t, x, u^0(t))$ is a matrix with nonnegative off-diagonal entries, and $\frac{\partial g}{\partial x}(t, x, u^0(t))$ is a nonnegative covector, i.e., there exists a number $d(t, x) \in \mathbb{R}$ such that the following relation holds:

$$\frac{\partial g}{\partial x}(t, x, u^0(t)) \succ 0_{\mathbb{X}}, \quad \frac{\partial f}{\partial x}(t, x, u^0(t)) \succ d(t, x) 1_{\mathbb{L}}. \quad (17)$$

Then, there exists a τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$, and for every such multiplier we have $\psi^0 \succ 0_{\mathbb{X}}$, and

$$\lambda^0 \limsup_{t \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} I_\xi(t) \succ \psi^0(0) \succ \lambda^0 \lim_{t \rightarrow \infty} I_0(t) \succ 0_{\mathbb{X}} \quad (18)$$

holds, and all limits in (18) are correctly defined and finite.

If $\lambda^0 > 0$ (for example, under (12)) and there exists a Lebesgue point $t^* \in \mathbb{T}$ for the control u^0 such that

$$\frac{\partial g}{\partial x}(t^*, x^0(t^*), u^0(t^*)) \succ 0_{\mathbb{L}},$$

we have $\psi^0|_{[0, t^*]} \succ 0_{\mathbb{X}}$; in particular, $\psi^0(0) \succ 0_{\mathbb{X}}$.

Remark 5 Assume that under conditions of Corollary 7 we can choose $d(t, x) \equiv 0$, and the integral

$$\int_0^t \frac{\partial g}{\partial x}(\vartheta, x^0(\vartheta), u^0(\vartheta)) d\vartheta$$

unboundedly increases as $t \rightarrow \infty$; then, all τ -vanishing solutions are degenerate.

Indeed, under $d(t, x) \equiv 0$, we can assume $r_0 \equiv 1$; then, in the case $\lambda^0 > 0$, (16) would, for $T = 0$, imply the boundedness of this integral.

Note that in [9, Theorem 1], [6, Theorem 10.1], [8, Theorem 5] the estimate $\psi \succcurlyeq 0_{\mathbb{X}}$ is made for problems

$$\dot{x} = f(x, u), u \in U, x(0) = x_0, \quad \int_0^\infty e^{-\rho t} g(x(t), u(t)) dt \rightarrow \max. \quad (19)$$

The most general case is examined in [8, Theorem 5]; namely, a variant of Corollary 7 is stated: if (17) is satisfied for all $t \in \mathbb{T}, u \in U(t), x \in \mathbb{X}$ (see [8, (A8)]), then $\psi \succcurlyeq 0_{\mathbb{X}}$, and estimate (18) holds (see [8, (5.5)]); the conditions, under which $\psi \succ 0_{\mathbb{X}}$ holds in addition to the above, are also specified. The explicit form of estimate (18) under the very strong conditions on f and g is also specified in [47, (23)–(26)].

Let us also remark that in all papers mentioned, the nondegeneracy of the problem was not assumed (and was not directly reduced to inequality (12)), it had to be proved; for example in [8, Theorem 5], it is demonstrated with the aid of the stationarity condition from additional proposition [8, (A7)]: on any admissible trajectory, there are some (t, u) , for which $f(x(t), u) \succ 0_{\mathbb{X}}$.

5 Explicit form of τ -vanishing shadow price

Previously, we examined two transversality conditions (11a) and (11b); let us now consider the two conditions

$$\lim_{t \rightarrow \infty} \|\psi^0(t)A_0(t)\|_{\mathbb{X}} = 0, \quad (20a)$$

$$\liminf_{n \rightarrow \infty} \|\psi^0(\tau_n)A_0(\tau_n)\|_{\mathbb{X}} = 0. \quad (20b)$$

Lemma 1 *For each solution $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{D}$, the transversality condition (20b) holds iff $\psi^0(0)$ is a partial limit of the sequence $(\lambda^0 I_0(\tau_n))_{n \in \mathbb{N}}$.*

Passing to the limit in $\lambda^0 I_0(\tau_n) = \lambda^0 (I_0(\tau_n) - I_0(0)) = \psi^0(0)A_0(0) - \psi^0(\tau_n)A_0(\tau_n)$, we obtain what was required; $\lambda^0 \neq 0$ by virtue of (5a).

Lemma 2 *If a nontrivial Lagrange multiplier $(1, \psi^0)$ associated with (x^0, u^0) satisfies the transversality condition (20b), then this multiplier is τ -vanishing.*

Indeed, there exists $\tau' \subset \tau$, for which $\psi^0(\tau'_n)A_0(\tau'_n) \rightarrow 0_{\mathbb{X}}$. Then $\psi^0(0) - I_0(\tau'_n) = \psi^0(\tau'_n)A_0(\tau'_n) \rightarrow 0_{\mathbb{X}}$, and $I_0(\tau'_n) \rightarrow \psi^0(0)$. Set $\psi_n(t) \triangleq (I_0(\tau'_n) - I_0(t))A_0^{-1}(t)$. Then $\psi_n(\tau'_n) = 0_{\mathbb{X}}$, $\psi^0(0) - \psi_n(0) = \psi^0(0) - I_0(\tau'_n) \stackrel{(6d)}{=} \psi^0(\tau'_n)A_0(\tau'_n) \rightarrow 0_{\mathbb{X}}$. The proof is completed by virtue of the uniform on each compact convergence $\psi_n \rightarrow \psi^0$.

5.1 Uniformity in initial conditions.

Theorem 3 *Assume that conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. Let one of the two conditions*

$$\text{either } \exists I_* \triangleq \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} I_\xi(\tau_n) \in \mathbb{X}; \quad (21a)$$

$$\text{or } \exists \iota_* \triangleq \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \frac{I_\xi(\tau_n)}{\|I_\xi(\tau_n)\|_{\mathbb{X}}} \in \mathbb{X}, \quad \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \|I_\xi(\tau_n)\|_{\mathbb{X}} = \infty \quad (21b)$$

hold.

Then, there exists a τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$. Moreover, this multiplier satisfies for all $T \in \mathbb{T}$ the corresponding formula of

$$\lambda^0 = 1, \quad \psi^0(T) \triangleq \left(I_* - \int_0^T \frac{\partial g}{\partial x}(t, x^0(t), u^0(t)) A_0(t) dt \right) A_0^{-1}(T); \quad (22a)$$

$$\lambda^0 = 0, \quad \psi^0(T) \triangleq \iota_* A_0^{-1}(T). \quad (22b)$$

Corollary 8 Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. Let the limit

$$\lim_{t \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} I_\xi(t) = \int_0^\infty \frac{\partial g}{\partial x}(t, x^0(t), u^0(t)) A_0(t) dt$$

be well-defined and finite.

Then, the pair (x^0, u^0) is normal and there exists a unique τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$. Moreover, for every solution $(x^0, u^0, \lambda^0, \psi^0)$ of core relations of the Maximum Principle (4a)–(4c) and (5b), the following conditions are equivalent:

- 1) its Lagrange multiplier (λ^0, ψ^0) is τ -vanishing;
- 2) the transversality condition (20b) holds;
- 3) the transversality condition (20a) holds;

$$4) \quad \lambda^0 \triangleq 1, \quad \psi^0(T) \triangleq \int_T^\infty \frac{\partial g}{\partial x}(t, x^0(t), u^0(t)) A_0(t) dt A_0^{-1}(T) \quad \forall T \in \mathbb{T}. \quad (22c)$$

Case (b) of Theorem 3 is shown in Corollary 6, case (a) will be proved below together with Proposition 5.

In contrast with (a), case (b) expresses the τ -vanishing Lagrange multiplier of a degenerate problem; the author has no knowledge of similar results. Together, these two cases allow to solve problem (1a)–(1b) through relations of the Maximum Principle regardless of its degeneracy (see, for example, Example 3).

The alternative (21a) \Rightarrow (22a) vs (21b) \Rightarrow (22b) is sufficiently convenient. The need for existence of the limit as $n \rightarrow \infty$ in one of relations (21a), (21b) can always be satisfied if we consider a subsequence. However, Example 4 shows that a unique τ -vanishing multiplier does not necessarily satisfy (20b), even for normal problems.

Then, the limit in (21a) (or (21b)) should exist not only for $\xi = 0_{\mathbb{X}}$, but also as $\xi \rightarrow 0_{\mathbb{X}}$. In some cases it is provided outright, for example, if the functions f and g are linear by x (see Example 3), or (see Example 5) by the following remark:

Corollary 9 Assumptions of Theorem 3 hold for a subsequence $\tau' \subset \tau$ if one of the assumptions either the functions f, g are linear with respect to x ,

$$\begin{aligned} \text{or} \quad & \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} (I_\xi(\tau_n) - I_0(\tau_n)) = 0_{\mathbb{X}}, \\ \text{or} \quad & \lim_{n \rightarrow \infty, \xi \rightarrow 0_{\mathbb{X}}} \frac{I_\xi(\tau_n) - I_0(\tau_n)}{\|I_0(\tau_n)\|_{\mathbb{X}}} = 0_{\mathbb{X}}, \end{aligned}$$

is satisfied.

Let us finish the proof of Theorem 3. Substituting $T = 0$ into (22a) yields $I_* = \psi^0(0)$; then, Lemma 1 implies

Lemma 3 A solution (x^0, ψ^0) of (4a)–(4b) given by formula (22a) satisfies (20b) iff I_* is a partial limit of the sequence $(\lambda^0 I_0(\tau_n))_{n \in \mathbb{N}}$,

Proposition 5 Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. Let the map I_0 be bounded and let

$$\lim_{\xi \rightarrow 0_x} \|I_\xi - I_0\|_C = 0.$$

Then, the pair (x^0, u^0) is normal and

- 1) there exists a τ -vanishing multiplier $(1, \psi^0) \in \Lambda$ such that transversality condition (20b) holds;
- 2) a Lagrange multiplier (λ^0, ψ^0) associated with (x^0, u^0) is τ -vanishing iff the transversality condition (20b) holds.
- 3) a limit point $I_* \in \mathbb{X}$ of the sequence $(I_0(\tau_n))_{n \in \mathbb{N}}$ corresponds to each τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$, and a τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$ corresponds to each limit point $I_* \in \mathbb{X}$ of the sequence $(I_0(\tau_n))_{n \in \mathbb{N}}$. This bijection is given by (22a).

Proof. By Theorem 2, a τ -vanishing multiplier exists; by Remark 3, any τ -vanishing multiplier (λ^0, ψ^0) satisfies $\lambda^0 > 0$; moreover, by (5b), if $(\lambda^0, \psi^0) \in \Lambda$, then $\lambda^0 = 1$. Now, by (9a), we have

$$\psi^0(0) = \lim_{n \rightarrow \infty} \lambda^n I_{\xi_n}(\tau'_n) = \lambda^0 \lim_{n \rightarrow \infty, \xi \rightarrow 0_x} I_\xi(\tau'_n) = \lambda^0 I_*,$$

and from Lemmas 1 and 3, we obtain (20b) and (22a). The inverse is true by virtue of Lemma 2. \square

5.2 Uniformity by control

Formulations of the preceding section can be expressed in another form. By varying, instead of the initial point ξ , the control u near u^0 , we pass from x_ξ, A_ξ, I_ξ to x^u, A^u, I^u .

Fix pair $(p, \nu) \in (0, \infty) \times B_{loc}(\mathbb{T}, \mathbb{R}_{>0})$. As in Remark 3, we have

Corollary 10 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$. If for the control u^0 and some subsequence $\tau' \subset \tau$ we have

$$\limsup_{n \rightarrow \infty, \varrho(\eta, u^0; \tilde{\mathcal{L}}_\nu^p([0, \tau'_n], \mathbb{U})) \rightarrow 0} \left\| \int_0^{\tau'_n} \frac{\partial g}{\partial x}(t, x^\eta(t), u(t)) A^\eta(t) dt \right\|_{\mathbb{X}} < \infty,$$

then the pair (x^0, u^0) is normal; there exists a τ -vanishing multiplier $(1, \psi) \in \Lambda$.

Proof. By Remark 2, there exist a τ' -vanishing multiplier (λ^0, ψ^0) and sequences $\tau'' \subset \tau', (x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbb{N}}$ such that Remark 1 and $\varrho(\eta^n, u^0; \tilde{\mathcal{L}}_\nu^p(\mathbb{T}, \mathbb{U})) \rightarrow 0$ hold. Then, $\varrho(\eta^n, u^0; \tilde{\mathcal{L}}_\nu^p([0, \tau_n''], \mathbb{U})) \rightarrow 0$; therefore, $(I^{\eta^n}(\tau_n''))_{n \in \mathbb{N}}$ is bounded by the assumption of the corollary. But $\lambda^n I^{\eta^n}(\tau_n'') \rightarrow \psi^0(0)$, thus $\lambda^0 > 0$. Now $(1, \psi^0/\lambda^0)$ is a τ -vanishing multiplier. \square

Corollary 11 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$ hold. Let I_0 be bounded and let

$$\|I_0 - I^\eta\|_{C([0, \tau_n], \mathbb{X})} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \varrho(\eta, u^0; \tilde{\mathcal{L}}_\nu^p([0, \tau_n], \mathbb{U})) \rightarrow 0.$$

Then, the pair (x^0, u^0) is normal, and

1) a τ -vanishing multiplier $(1, \psi^0) \in \Lambda$ corresponds to each partial limit $I_* \in \mathbb{X}$ of the sequence $(I_0(\tau_n))_{n \in \mathbb{N}}$ by formula (22a);

2) all such multipliers satisfy transversality condition (20b).

Proof. Let I_* be the limit of $(I_0(\tau'_n))_{n \in \mathbb{N}}$ for certain $\tau' \subset \tau$. Then, by Corollary 10, there exists a τ' -vanishing multiplier $(1, \psi^0)$ such that $\psi^0(0) = \lim_{n \rightarrow \infty} I^{\eta^n}(\tau''_n)$ for some $\tau'' \subset \tau'$. By the assumption of the corollary this, limit corresponds with I_* , i.e., $\psi^0(0) = \lambda^0 I_*$. But this, by Lemma 1, is equivalent to (20b). Substituting $\psi^0(0) = \lambda^0 I_*$ into (6d), we obtain (22a). \square

Repeating the proof of Corollary 10, but, this time, using (13), we have

Corollary 12 Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$ hold. Let for some $\iota_* \in \mathbb{X}$ there be

$$\frac{I^\eta(\tau_n)}{\|I^\eta(\tau_n)\|_{\mathbb{X}}} \rightarrow \iota_*, \quad \|I^\eta\|_{C([0, \tau_n], \mathbb{X})} \rightarrow \infty \quad \text{as } n \rightarrow \infty, \varrho(\eta, u^0; \tilde{\mathcal{L}}_\nu^p([0, \tau_n], \mathbb{U})) \rightarrow 0.$$

Then, for the pair (x^0, u^0) , there exists a degenerate τ -vanishing multiplier $(0, \psi^0)$ such that condition (20b) and formula (22b) hold.

5.3 Conditions guaranteeing convergence to I_* .

Let us consider the conditions on the system that are both sufficiently easy to check and sufficient to make use of Corollary 8.

Proposition 6 Assume conditions $(\mathbf{u}_\sigma), (\mathbf{fg}), (\tau)$ hold. For certain measurable functions $F \in \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{L})$, $G \in \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{X})$, a summable function $\omega \in \mathfrak{L}^1(\mathbb{T}, \mathbb{T})$, let

$$G(t) \succcurlyeq \frac{\partial g}{\partial x}(t, x, u^0(t)) \succcurlyeq -G(t), \quad F(t) \succcurlyeq \frac{\partial f}{\partial x}(t, x, u^0(t)) \succcurlyeq -F(t), \quad (23a)$$

$$\|G(t)B_*(t)\|_{\mathbb{X}} \leq \omega(t) \quad (23b)$$

for all $(t, x) \in \mathbb{T} \times \mathbb{X}$, where B_* is a matrix solution of

$$\dot{B}_*(t) = F(t)B_*(t), \quad B_*(0) = 1_{\mathbb{L}} \quad \forall a.a. t \in \mathbb{T}. \quad (23c)$$

Then, the result of Corollary 8 holds.

Proof. For each $B = (b_{ij})_{i,j \in \overline{1,m}} \in \mathbb{L}$, $C = (c_i)_{i \in \overline{1,m}} \in \mathbb{X}$, let us introduce

$$B^\sharp \triangleq (|b_{ij}|)_{i,j \in \overline{1,m}} \in \mathbb{L}, \quad C^\sharp \triangleq (|c_i|)_{i \in \overline{1,m}} \in \mathbb{X}.$$

It is easy to see that $B^\sharp \succcurlyeq 0_{\mathbb{L}}$, $C^\sharp \succcurlyeq 0_{\mathbb{X}}$, $B^\sharp \succcurlyeq B \succcurlyeq -B^\sharp$, $C^\sharp \succcurlyeq C \succcurlyeq -C^\sharp$. Moreover, $C^\sharp B^\sharp \succcurlyeq CB \succcurlyeq -C^\sharp B^\sharp$ for all $B \in \mathbb{L}, C \in \mathbb{X}$.

Denote by $F_\xi(t)$ the matrix $\frac{\partial f}{\partial x}(t, x_\xi(t), u^0(t))$ for all $t \in \mathbb{T}$. Now, for all $\xi \in \mathbb{X}$, we have

$$F(t) \succcurlyeq F_\xi^\sharp(t) \succcurlyeq F_\xi(t) \succcurlyeq -F_\xi^\sharp(t) \succcurlyeq -F(t) \quad \forall a.a. t \in \mathbb{T};$$

comparing the right-hand sides and the initial conditions of equations (23c), (6b), and equation

$$\dot{B}_\xi(t) = F_\xi^\sharp(t)B_\xi(t), \quad B_\xi(0) = 1_{\mathbb{L}},$$

for its solution B_ξ by the comparison theorem, we obtain

$$B_*(t) \succ B_\xi(t) \succ A_\xi(t) \succ -B_\xi(t) \succ -B_*(t) \quad \forall \text{ a.a. } t \in \mathbb{T};$$

in particular, $B_*(t) \succ A_\xi^\sharp(t)$. Now, we have $G(t)B_*(t) \succ \left(\frac{\partial g}{\partial x}(t, x_\xi(t), u^0(t))\right)^\sharp A_\xi^\sharp(t) \succ \left(\dot{I}_\xi(t)\right)^\sharp$, whence we obtain $G(t)B_*(t) \succ \dot{I}_\xi(t) \succ -G(t)B_*(t)$, $\|\dot{I}_\xi(t)\|_{\mathbb{X}} \leq \|G(t)B_*(t)\|_{\mathbb{X}} \leq \omega(t)$ for all $\xi \in \varepsilon_0\mathbb{D}$, for almost all $t \in \mathbb{T}$. We have

$$\begin{aligned} \|I_\xi\|_C &\leq \|I_\xi\|_{C([0,T],\mathbb{X})} + \int_T^\infty \omega(t) dt, \\ \|I_\xi - I_0\|_C &\leq \|I_\xi - I_0\|_{C([0,T],\mathbb{X})} + 2 \int_T^\infty \omega(t) dt. \end{aligned}$$

For each $\varepsilon > 0$, it is possible to find $T \in \mathbb{T}$, for which the second summands do not exceed ε , and yet $I_\xi|_{[0,T]} \rightarrow I_0|_{[0,T]}$ for $\xi \rightarrow 0_{\mathbb{X}}$. Then all conditions of Corollary 8 hold. \square

Remark 6 The first condition of (23a) of Proposition 6 could be formally weakened down to

$$F(t) + m(t)1_{\mathbb{L}} \succ \frac{\partial f}{\partial x}(t, x, u^0(t)) \succ -F(t) - m(t)1_{\mathbb{L}},$$

for some summable function $m \in \mathfrak{L}^1(\mathbb{T}, \mathbb{T})$.

Indeed, consider a number $R = e^{\int_0^\infty m(\theta)d\theta} \in \mathbb{T}$, a summable function $\omega_1 \triangleq R\omega$, and a matrix function $F_1 \triangleq F + m1_{\mathbb{L}}$. Now, $B_1(t) \triangleq e^{\int_{[0,t]} m(\theta)d\theta} B_*(t)$ solves the equation $\dot{B}_1 = F_1 B_1$, $B_1(0) = 1_{\mathbb{L}}$ and

$$\|G(t)B_1(t)\|_{\mathbb{X}} = e^{\int_{[0,t]} m(\theta)d\theta} \|G(t)B_*(t)\|_{\mathbb{X}} \leq e^{\int_{[0,t]} m(\theta)d\theta} \omega(t) \leq R\omega(t) = \omega_1(t).$$

Thus, under conditions of the remark, all propositions of Proposition 6 hold for F_1, ω_1 in the place of F, ω .

Note that conditions of Proposition 6 (taking into account Remark 6) for a smooth control problem without phase restrictions are weaker than conditions [38, (C1)-(C3)]. To be more precise, condition [38, (C1)] is exactly condition **(u)**, and [38, (C2)] is exactly (23b). Condition [38, (C3)] requires $\|G(t)B_*(t)B_*^{-1}(\theta)\|_{\mathbb{X}} \leq \omega(t)$ for all $t \in \mathbb{T}, \theta \in [0, t]$, while condition (23a) requires this only for $t \in \mathbb{T}, \theta = 0$. In particular, in [6, Example 16.1], conditions of [6, Theorem 12.1] and Proposition 6 hold if $\rho > 0$, and conditions [38, (C1)-(C3)] only hold if $\rho > 1$.

Corollary 13 Assume conditions **(u)**, **(fg)**, (τ) hold. For a summable function $\omega \in \mathfrak{L}^1(\mathbb{T}, \mathbb{T})$ for all $u \in \mathfrak{U}$, let

$$\left\| \frac{\partial g}{\partial x}(t, x^u(t), u(t)) A^u(t) \right\|_{\mathbb{X}} \leq \omega(t). \quad (24)$$

Then, the pair (x^0, u^0) is normal and Corollary 8 holds with exception of uniqueness of the τ -vanishing multipliers; specifically,

- 1) exactly one τ -vanishing multiplier satisfies (5b) and (20b);
- 2) exactly one τ -vanishing multiplier satisfies (5b) and (20a);
- 3) actually, it is the τ -vanishing multiplier $(1, \psi^0) \in A$; and this multiplier could be obtained by formula (22c).

Proof. Note that (24) holds not only for all $u \in \mathfrak{U}$, but also for all $\eta \in \widetilde{\mathfrak{U}}$; then, for all $T \in \mathbb{T}$, we have

$$\begin{aligned} \|I^\eta\|_C &\stackrel{(24)}{\leq} \|I^\eta\|_{C([0,T],\mathbb{X})} + \int_T^\infty \omega(t) dt, \\ \|I^\eta - I_0\|_C &\stackrel{(24)}{\leq} \|I^\eta - I_0\|_{C([0,T],\mathbb{X})} + 2 \int_T^\infty \omega(t) dt. \end{aligned}$$

For each $\varepsilon > 0$ there exists a $T \in \mathbb{T}$ such that the second summands do not exceed $\varepsilon/2$. Let us construct the τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$ by a limit of sequences from Remark 1, but Proposition 1 implies $I^\eta|_{[0,T]} \rightarrow I_0|_{[0,T]}$ for $\eta \rightarrow \tilde{u}^0$. Hence, $\|I^{\eta^n} - I_0\|_C \rightarrow 0$ and I_0 is bounded. Since $\psi^n(\tau'_n) = 0_{\mathbb{X}}$, we know that (20b) holds for ψ^0 .

From (24) for $u = u^0$ we see that for any unboundedly increasing sequence of times v , the sequence $(I_0(v_n))_{n \in \mathbb{N}}$ is fundamental and thus it has the limit point I_* . Since this is correct for any unboundedly increasing sequence of times, $I_0(t) \rightarrow I_*$ as $t \rightarrow \infty$. Lemma 1 yields item 2). Finally, Lemma 3 implies (22c). \square

The formula (22c) was obtained by Kryazhimskii and Aseev under easily checked assumptions on growth of functions f, g and their derivaties (see stationary case in [6, Theorem 12.1], [8, Theorem 4] and non-stationary case in [10, Theorem 1]). This condition generalizes (see [6, Sect. 16], [10]) a number of transversality conditions; in particular, it is more general than the conditions that were obtained for linear systems in [11].

From conditions of [7, Theorem 2], [6, Theorem 12.1], and [5, Theorem 1] it follows that for some $\alpha, \beta > 0$ and for all admissible controls $u \in \mathfrak{U}$, all trajectories x , and all fundamental matrices A , the following inequality holds:

$$\left\| \frac{\partial g}{\partial x}(t, x(t), u(t)) \right\| \|A(t)\| \leq \beta e^{-\alpha t} \quad \forall t \in \mathbb{T} \quad (25)$$

(see, for example, [6, (A5)-(A7)]). This is stronger than the conditions of Corollary 13. In paper [10], it was actually assumed that (25) holds for $x = x_\xi, A = A_\xi, u = u^0$ if ξ is sufficiently small. This is slightly stronger than the stability condition in Corollary 8. However, it is worth noting that [10, Theorem 1] uses a more general definition of optimality (the locally weakly overtaking optimality). In addition, condition (25) can be verified by calculating the characteristic Lyapunov exponents of the system of the Maximum Principle, see [6, Sect. 12], [7, Sect. 3], [10, Sect. 5].

Observe that (25) are characteristic of economic problems with exponentially decreasing discount factor; however, one could consider other non-subexponential discount factors (see [23, 24, 48, 49]). Example 5 exhibits the solution of a problem with such discount factor.

For economic problems with decreasing discount factor (specifically, for (19)) in [8, Theorem 4], sufficiently broad conditions for applicability of formula (22c) were obtained. It turns out that it is sufficient to connect (see [8, (A4)] and (26)) the growth of I^u with the growth of J^u . In contrast with the results of [10] or Corollary 13, the finiteness of the optimal result on the optimal trajectory is required, and it is not guaranteed that the τ -vanishing multiplier is unique. Let us transfer this result of [8, Theorem 4] from case (19) to general non-stationary system (1a)–(1b).

Corollary 14 *Assume conditions $(\mathbf{u}), (\mathbf{fg}), (\tau)$ hold. Let there exist the finite limit $\lim_{n \rightarrow \infty} J^{u^0}(\tau_n)$. Let a functions $\omega_0, \omega_\infty \in C(\mathbb{T}, \mathbb{T})$ satisfy $\omega_0(0) = 0, \omega_\infty(\tau_n) \rightarrow 0$ as $n \rightarrow \infty$. For all $u \in \mathfrak{U}$ from some $\mathcal{L}_\nu^p(\mathbb{T}, \mathbb{U})$ -neighborhood*

O_ν^p of the control u^0 for all $k, n \in \mathbb{N}, k < n$, let there be

$$\left\| \int_{\tau_k}^{\tau_n} \frac{\partial g}{\partial x}(t, x^u(t), u(t)) A^u(t) dt \right\|_{\mathbb{X}} \leq \omega_\infty(\tau_k) + \omega_0(|J^u(\tau_n) - J^u(\tau_k)|). \quad (26)$$

Then, the pair (x^0, u^0) is normal, the limit $I_* = \lim_{n \rightarrow \infty} I^0(\tau_n) \in \mathbb{X}$ is well-defined, and

1) exactly one multiplier satisfies (5b) and (20b);

2) actually, it is the τ -vanishing multiplier $(1, \psi^0) \in \Lambda$; and this multiplier could be obtained by formula (22c).

Proof. There exists a sequence $(s_k)_{k \in \mathbb{N}} \downarrow 0$ such that for all $k, n \in \mathbb{N}, k < n$, we have $|J^{u^0}(\tau_n) - J^{u^0}(\tau_k)| < s_k$. Substituting $u = u^0$ into (26) yields the existence of the finite limit $I_* = \lim_{n \rightarrow \infty} I^0(\tau_n)$. Now, as in the proof of Corollary 13, we show that there exists the unique solution from \mathfrak{Y} that satisfies (20b) and that for it, accurately to a positive factor, the formula (22a) is correct. It only remains to prove that multiplier defined by (22a) is τ -vanishing.

By Theorem 2, for this problem there exists the τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$ that was constructed by the uniform limit of sequences $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbb{N}} \in \widetilde{\mathfrak{Y}}^{\mathbb{N}}$ from Remark 1. Passing to the subsequence $\tau' \subset \tau$ if necessary, we may assume $\eta^n \in cl O_\nu^p$ for all $n \in \mathbb{N}$, then $(\widetilde{26})$ hold for each η^n . The function ω_0 can be considered monotonic without loss of generality. Then, using the triangle inequality twice and, by the inequality $\widetilde{J}^{\eta^n}(\tau_n) - J^{u^0}(\tau_n) \geq 0$, for all $k, n \in \mathbb{N}, k < n$, we have the following:

$$\begin{aligned} & \|I^{\eta^n}(\tau_n) - I^0(\tau_n)\|_{\mathbb{X}} - \|I^{\eta^n}(\tau_k) - I^0(\tau_k)\|_{\mathbb{X}} \leq \\ & \|I^{\eta^n}(\tau_n) - I^{\eta^n}(\tau_k)\|_{\mathbb{X}} + \|I^0(\tau_n) - I^0(\tau_k)\|_{\mathbb{X}} \stackrel{(26)}{\leq} \\ & 2\omega_\infty(\tau_k) + \omega_0(|J^{u^0}(\tau_n) - J^{u^0}(\tau_k)|) + \omega_0(|\widetilde{J}^{\eta^n}(\tau_n) - \widetilde{J}^{\eta^n}(\tau_k)|) \leq \\ & 2\omega_\infty(\tau_k) + \omega_0(|J^{u^0}(\tau_n) - J^{u^0}(\tau_k)|) + \\ & + \omega_0(|\widetilde{J}^{\eta^n}(\tau_n) - J^{u^0}(\tau_n)| + |\widetilde{J}^{\eta^n}(\tau_k) - J^{u^0}(\tau_k)| + |J^{u^0}(\tau_n) - J^{u^0}(\tau_k)|) \stackrel{(2)}{\leq} \\ & 2\omega_\infty(\tau_k) + 2\omega_0(\gamma_n^2 + |\widetilde{J}^{\eta^n}(\tau_k) - J^{u^0}(\tau_k)| + s_k). \end{aligned}$$

Since $I^\eta, \widetilde{J}^\eta$ converges to I_0, J^{u^0} uniformly on any compact and by definitions of ω_0, ω_∞ and γ_n, r_n, s_k , passing to the limits first, as $n \rightarrow \infty$, and then, as $k \rightarrow \infty$, we see that

$$\limsup_{n \rightarrow \infty} \|I^{\eta^n}(\tau_n) - I^0(\tau_n)\|_{\mathbb{X}} \leq 2\omega_\infty(\tau_k) + 2\omega_0(s_k)$$

and $I^{\eta^n}(\tau_n) - I^0(\tau_n) \rightarrow 0_{\mathbb{X}}$. Now, by Remark 1, we have $\lambda^n I^0(\tau_n) \rightarrow \psi^0(0)$. Since $I^0(\tau_n) \rightarrow I_*$, we know that $\lambda^0 > 0$ and $\lambda^0 \psi^0(0) = I_*$ hold. By dividing this (λ^0, ψ^0) on λ^0 , we obtain (22a). \square

6 Examples

Example 1 The feature of [38, Ex. 10.2] lies in the fact that transversality condition (11a) fails to give any information that could help us in determining the unique Lagrange multiplier. Let us show that the definition of a τ -vanishing multiplier allows us to do it.

$$\dot{x} = ux, \quad x(0) = 1, \quad u \in [1/2, 1], \quad J^u(T) \triangleq \int_0^T x e^{-2t} dt \stackrel{T \rightarrow \infty}{\rightsquigarrow} \max.$$

Here, $H = u\psi x + e^{-2t}\lambda x$ and $\dot{\psi} = -u\psi - e^{-2t}\lambda$. Then, $A = x, I^u = J^u$; consider $F = 1, G = e^{-2t}, \omega(t) = e^{-t}$. By Proposition 6, there exists the unique τ -vanishing multiplier. Substituting it into H , we obtain $H(x^0(t), t, u^0(t), \lambda^0, \psi^0(t)) = u^0 \lambda^0 (J^{u^0}(\infty) - J^{u^0}(t)) + e^{-2t} \lambda x^0(t)$; now, from (4c), we have $u^0 \equiv 1, J^{u^0}(+\infty) = 1$; then, $\psi^0(0) = \lambda^0 = 1$, it is a unique τ -vanishing multiplier. (Of course, in this example, the control u^0 is easily found in view of the monotonicity of f, g and Corollary 7).

The alternative (21a) \Rightarrow (22a) versus (21b) \Rightarrow (22b) allows us to effectively reduce an optimal problem to the boundary problem of relations of the Maximum Principle. The only obstacle is the uniformity of limits in (21a) and (21b). In some cases, the uniformity of these limits is trivial, for example, when the functions f and g are linear by x . Thus, such problems are easy to solve. Let us demonstrate this by the following example:

Example 2

$$\dot{x} = y, \quad \dot{y} = -x + u, \quad x(0) = 1, \quad y(0) = 0, \quad u \in [-1, 1], \quad \int_0^T y dt \xrightarrow{T \rightarrow \infty} \max$$

Here, for all $t, T, s \in \mathbb{T}, \xi \in \mathbb{X}$, we have

$$A_\xi(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, I_\xi(T) = (\cos T - 1, \sin T),$$

$$I_0(s)A_0^{-1}(T) = (\cos(s - T) - \cos T, \sin(s - T) + \sin T).$$

Now, because I_ξ is 2π -periodic, for any sequence $(\tau_n)_{n \in \mathbb{N}}$ there exists a $\varsigma \in [0, 2\pi]$ and subsequence $\tau' \subset \tau$ such that $I_{\xi}(\tau'_n) \rightarrow I_0(\varsigma)$, whence, by Theorem 3,

$$\begin{aligned} \psi^0(T) &= (I_0(\varsigma) - I_0(T))A_0^{-1}(T) = (\cos(\varsigma - T) - 1, \sin(\varsigma - T)); \\ u^0(T) &= \arg \max_{u \in [-1, 1]} (\cos(\varsigma - T) - 1, \sin(\varsigma - T)) \begin{pmatrix} 0 \\ u \end{pmatrix} = \arg \max_{u \in [0, 1]} \sin(\varsigma - T)u, \text{ i.e.} \\ u^0(T) &= \operatorname{sgn} \sin(\varsigma - T) \quad \forall \text{ a.a. } T \in \mathbb{T}. \end{aligned} \tag{27}$$

Observe that the proposed approach finds, first of all, τ -optimal controls. Indeed, let the sequence τ be given. Express each τ_n in the form $\tau_n = 2\pi k_n + \sigma_n$, where $\sigma_n \in [0, 2\pi)$. Substituting each limit point ς of the sequence $(\sigma_n)_{n \in \mathbb{N}}$ into (22a) yields all corresponding τ -vanishing multipliers; moreover, formula (27) yields all prospective τ -optimal controls.

It is easy to check (see [43]) that any control of form (27) is uniformly weakly overtaking optimal, thus each of them is τ -optimal for its sequence τ .

Also observe that this example specifies why it is impossible to replace transversality condition (20b) in Proposition 5 with the stronger one (20a).

Example 3 Theorem 3 allows, in some circumstances, to find optimal solutions for degenerate problems in the way it is done for nondegenerate. Let us show this. Consider the modification of the well-known Halkin's example [28] (see also [35, Ex. 5.1], [9, Ex. 1])

$$\dot{x} = ux, \quad x(0) = 1, \quad \int_0^T (1 - u)x dt \xrightarrow{T \rightarrow \infty} \max, \quad u \in [\alpha, \beta] \quad (\alpha \leq \beta).$$

Let there exist a weakly uniformly overtaking optimal control in this problem, then, for some sequence τ , this control is τ -optimal.

Here, $A_\xi(T) = x^0(T)$ and $I_\xi(T) = J^{u^0}(T)$. Passing, if necessary, from τ to its subsequence, we face one of the three cases:

A) $J^{u^0}(\tau_n) \rightarrow +\infty$. From Theorem 3 (b) $\iota_* = 1$, $\lambda = 0$, $H(T) = u^0$, $u^0 \equiv \beta$; if we substitute this into $J^{u^0}(T)$, we will obtain $0 \leq \beta < 1$.

B) $J^{u^0}(\tau_n) \rightarrow -\infty$; similarly, we have $u^0 \equiv \alpha > 1$.

C) $J^{u^0}(\tau_n) \rightarrow I_* \in \mathbb{R}$. Here, by Theorem 3 (a), from (21a) follows (22a). Consider $R(t) \triangleq I_* - J^{u^0}(t) - x^0(t)e^{\rho t}$. Now we have $H(t) = R(t)u - x$, and $u^0(t)$ is defined by the sign of $R(t)$. Since $\dot{R}(t) = -x(t) < 0$, there is at most one switching point.

Note that $u(t) = \gamma$ for all $t > T$, and for some $T \in \mathbb{T}$, $\gamma \in [\alpha, \beta]$. The boundedness of $I_* - J^{u^0}(t)$ provides that either $\gamma < 0$ or $\gamma = 1$. We claim that the sign of $R(t)$ does not change. Assume the converse, and let there be a switching point $T > 0$; then, $R(t) < 0$ for $t > T$, and $u(t) = \beta = 1$, whence $I_* = J^{u^0}(T)$, i.e., $x(T) = -R(T) = 0$, which is impossible. Hence, if $R(0) > 0$, then $u^0 \equiv \beta = \gamma < 0$; else, $u^0 \equiv \alpha = \gamma = 1$.

Checking this, we show that $u^0 \equiv \alpha$ for $\alpha \geq 1$ and $u^0 \equiv \beta$ for $\beta < 1$ are indeed τ -optimal (moreover, even uniformly overtaking optimal) control in this problem. Consequentially, the problem has no τ -optimal (and, therefore, no weakly uniformly overtaking optimal) control if $\alpha < 1 \leq \beta$. On the other hand, in case $[\alpha, \beta] \triangleq [0, 1]$, the control $u^0 \equiv 0$ is decision horizon optimal (DH-optimal, see [14]). Therefore, in Theorem 2, we could not replace the τ -optimality (weakly uniformly overtaking optimality, uniformly overtaking optimality) with the *DH*-optimality (weekly agreeable, agreeable optimality; [14]).

Example 4 Consider the Arnold's model from [2]

$$\dot{x} = u, \quad x(0) = x_{**}, \quad u \in [1, 2], \quad x \in \mathbb{R}, \quad \frac{\int_0^T g(x) dt}{T} \xrightarrow{T \rightarrow \infty} \max,$$

where profit density, denoted by g , is a scalar 1-periodic smooth function with a finite number of critical points.

As shown in [3, 20], this problem has a unique periodic optimal solution u^0 , and for certain $g_* \in \mathbb{T}$, we have

$$(g(x^0(t)) < g_*) \Rightarrow (u^0(t) = 2) \quad (g(x^0(t)) > g_*) \Rightarrow (u^0(t) = 1) \quad \text{for a.a. } t \in \mathbb{T}. \quad (28)$$

Denote the period of this solution by T_0 .

Consider the sequence $\tau \triangleq (nT_0)_{n \in \mathbb{N}}$. Note that only the control u^0 is τ -optimal for the problem

$$\dot{x} = u, \quad x(0) = x_{**}, \quad u \in [1, 2], \quad x \in \mathbb{R}, \quad (29)$$

$$J(T) = \int_0^T g(x(t)) dt \xrightarrow{T \rightarrow \infty} \max.$$

Actually, it is possible to prove that this control is at most weakly uniformly overtaking and there are no other weakly uniformly overtaking optimal controls in this problem.

Application of Theorem 1 to problem (29) yields (28). Simple reflections on optimality show that $\min_{x \in [0, 1]} g(x) < g^* < \max_{x \in [0, 1]} g(x)$, by (4c) we have $\lambda > 0$ for any Lagrange multiplier (λ, ψ) associated

with (x^0, u^0) . However, no additional conditions on g_* could be obtained from the core relations of the Maximum Principle. Let us see if it is possible to do that using the approach of this paper.

It is obvious that $A_\xi \equiv 1_{\mathbb{L}}$. It is also easy to see that, using the substitution $\vartheta(t) = x^0(t), t = \vartheta^{-1}(x^0(t))$, we could obtain for all $T \in \mathbb{T}$ the following relation:

$$I_0(T) = \int_0^T \frac{dg}{dx}(x^0(t)) dt = \int_{x^0(0)}^{x^0(T)} \frac{dg(\vartheta)}{d\vartheta} \frac{d\vartheta}{u^0(t)};$$

now, if u^0 is constant on some interval (t_2, t_1) , then

$$I_0(t_1) - I_0(t_2) = \int_{x^0(t_2)}^{x^0(t_1)} \frac{dg(\vartheta)}{d\vartheta} \frac{d\vartheta}{u^0(t)} = \frac{g(x^0(t_1)) - g(x^0(t_2))}{u^0(\frac{t_1+t_2}{2})}.$$

If t_1, t_2 are switching points, then $g(x^0(t_1)) = g(x^0(t_2)) = g_*$, $I_0(t_1) = I_0(t_2)$. Since u^0 is T_0 -periodic, this immediately yields that the functions $x^0, g \circ x^0, I^0$ are also T_0 -periodic.

Observe that the τ -vanishing multiplier $(1, \psi^0) \in \Lambda$ exists. Let us show that it does not necessarily satisfy (20b) and (22c). Since I_0 is T_0 -periodic, $I_0(\tau_n) \equiv I_0(0)$, whence $I_* = 0$. If (22c) holds, then, for all $T \in \mathbb{T}$, we have $\psi^0(T) \triangleq -I_0(T)$. Substitution into the Hamiltonian yields $\mathcal{H}(t, x^0(t), u, 1, \psi^0(t)) = -I_0(t)u + g(x^0(t))$. Now (4c) implies that $u^0(t)$ is determined by the sign of $-I_0(t)$, whence $g_* = g(x^0(0)) = g(x_{**})$. But g_* is independent of the choice of the initial point on the cycle in auxiliary problem (29). Therefore, for a.a. x_{**} , formula (22a) is invalid in this problem. This trivially implies that a τ -vanishing control does not necessarily satisfy (20b), even for normal problems.

Is it possible to use the formula (22a) to find τ -vanishing multipliers in this problem? Strange as it sounds, yes.

Observe that the notion of τ -vanishing multiplier, as well as the core relations of the Maximum Principle (see [1]), is invariant with respect to coordinate transformations. Let us maximize $\bar{J}^u(T) = \ln(1 + J^u(T))$ instead of $J^u(T)$. Consider the problem

$$\dot{x} = u, \quad x(0) = x_{**}, \quad \dot{y} = g(x), \quad y(0) = 1, \quad u \in [1, 2], \quad (30)$$

$$\bar{J}(T) = \ln(1 + J(T)) = \ln y(T) = \int_0^T \frac{g(x(t))}{y(t)} dt \stackrel{T \rightarrow \infty}{\rightsquigarrow} \max.$$

Take an arbitrary control u^0 of form (28), and let its period be some T_0 . It is easily seen that $\bar{A}_\xi \equiv \begin{pmatrix} 1 & 0 \\ I_\xi & 1 \end{pmatrix}$,

$$\begin{aligned} \bar{I}_\xi(nT_0) &= \int_0^{nT_0} \frac{1}{y_\xi(t)} \left(\frac{dg(x_\xi(t))}{dx}, -\frac{g(x_\xi(t))}{y_\xi(t)} \right) \begin{pmatrix} 1 & 0 \\ I_\xi & 1 \end{pmatrix} dt = \\ &= \int_0^{nT_0} \frac{1}{y_\xi^2(t)} (\dot{I}_\xi(t)y_\xi(t) - I_\xi(t)\dot{y}_\xi(t), -\dot{y}_\xi(t)) dt = \\ \frac{(I_\xi(t), 1)}{y_\xi(t)} \Big|_{t=0}^{t=nT_0} &= \frac{(nI_\xi(T_0), 1)}{n(y_\xi(T_0) - y_\xi(0)) + y_\xi(0)} - (0, 1) \rightarrow \left(\frac{I_\xi(T_0)}{y_\xi(T_0) - y_\xi(0)}, -1 \right). \end{aligned}$$

Now, the theorem of continuous dependence on initial conditions implies (21a). Thus, Theorem 3 also holds for problem (30) for each control u^0 of form (28), and its proper τ -vanishing Lagrange multiplier is given by formula

(22a). Thus, formula (22a), under proper coordinate transformation, can be used to solve problem (29),(30), although this yields no additional conditions in comparison with the core relations of the Maximum Principle.

Actually, this is rather reasonable since a control of form (28) is weakly uniformly overtaking optimal for the objective functional

$$\bar{J}(T) \triangleq \ln(1 + \ln(1 + J(T))) = \int_0^T \frac{g(x)}{y(1 + \ln y)} dt \xrightarrow{T \rightarrow \infty} \max.$$

Therefore, in this problem, it has a τ -vanishing multiplier; since the definition of τ -vanishing multiplier is invariant, each control of form (28) has such a multiplier in problems (29) and (30) too even if the corresponding controls are not weakly uniformly overtaking optimal in these problems.

Let us show the example of reducing an infinite horizon optimal control problem to the boundary problem.

Example 5 In [10], the following stylized microeconomic problem was considered:

$$\begin{aligned} \dot{x}(t) &= -\nu x(t) + u(t), \quad x(0) = K_0, \quad u \geq 0; \\ J^u(T) &= \int_0^T e^{-dt} \left[e^{pt} (x(t))^\sigma - \frac{b}{2} (u(t))^2 \right] dt \xrightarrow{T \rightarrow \infty} \max. \end{aligned}$$

Here, $u(t)$ is the investment, $\nu \geq 0$ is the depreciation rate, $K_0 > 0$ is the given initial capital stock, e^{-dt} is the discount factor ($d \geq 0$), $e^{pt} \geq 0$ is the (exogenous) factor of technological advancement ($p \geq 0$), $bu^2(t)$ ($b > 0$) is the cost of investment $u(t)$, and $\sigma \in (0, 1]$ defines the production function. Under the assumption $d + \nu > \frac{p}{2-\sigma}$, it is shown that there are no optimal solutions for $p > d + \nu$, and, for $p < d + \nu$, each locally weakly overtaking control induces a solution of the boundary problem (see [10]).

Consider the following objective functional:

$$J^u(T) = \int_0^T g(t) (x(t))^\sigma - h(t) \frac{b}{2} (u(t))^2 dt \xrightarrow{T \rightarrow \infty} \max.$$

Here, $h(t)$ is the discount factor, $g(t)$ is the product of the discount factor and the factor of technological advancement.

Suppose that there exists a weakly overtaking optimal control u^0 . Then, for some sequence $\tau \uparrow \infty$, this solution is τ -optimal. Hence, there exists a τ -vanishing multiplier $(\lambda^0, \psi^0) \in \Lambda$.

Now, for all $\xi \in \mathbb{X}$, we have $A_\xi = e^{-\nu t}$,

$$I_\xi(T) = \int_0^T g(t) \sigma x_\xi^{\sigma-1}(t) e^{-\nu t} dt = \sigma \int_0^T g(t) e^{-\nu t} x_\xi^{\sigma-1}(t) dt.$$

Note that $x_\xi(t) - x^0(t) = \xi e^{-\nu t}$; now we have

$$I_\xi(T) - I_0(T) = \sigma \int_0^T g(t) e^{-\sigma \nu t} \left[(x^0(t) e^{\nu t} + \xi)^{\sigma-1} - (x^0(t) e^{\nu t})^{\sigma-1} \right] dt.$$

It is easy to see that $|(r + \xi)^{\sigma-1} - r^{\sigma-1}| \leq (2^{2-\sigma} - 2)|\xi| r^{\sigma-2} \leq (2^{2-\sigma} - 2)K_0 |\xi| r^{\sigma-1}$ if $2|\xi| < K_0 \leq r$. Since the function $x^0(t) e^{\nu t}$ is monotonically increasing, we obtain

$$|I_\xi(T) - I_0(T)| \leq \left| \int_0^T g(t) e^{-\nu t} (x^0)^{\sigma-1} dt \right| (2^{2-\sigma} - 2) K_0 |\xi| = |I_0(T)| (2^{2-\sigma} - 2) K_0 |\xi|$$

for all $T \in \mathbb{T}$, $2|\xi| < K_0$. Now, by Corollary 9, considering the subsequence if necessary, we have the conclusion of Theorem 3.

We claim that $(I_0(\tau_n))_{n \in \mathbb{N}}$ is bounded. Assume the converse; then, considering the subsequence if necessary, we come to (21b) and (22b), whence $\lim_{\xi \rightarrow 0, n \rightarrow \infty} I_\xi(\tau_n) = \pm\infty$, now $\iota^* = \pm 1$, $\lambda^0 = 0$ and by (4c) we have

$$u^0(t) = \arg \max_{u \in \mathbb{R}_{\geq 0}} e^{\nu t} I_0(t)(u - \nu x) = \arg \max_{u \in \mathbb{R}_{\geq 0}} I_0(t)u = \pm\infty,$$

which is impossible. This contradiction proves the boundedness of sequence $(I_0(\tau_n))_{n \in \mathbb{N}}$.

Now there exists a finite limit I_* of $(I_0(\tau'_n))_{n \in \mathbb{N}}$ for some $\tau' \subset \tau$. By Theorem 3, we have (22a), $\lambda^0 = 1$, $\psi^0(T) = (I_* - I_0(t))e^{\nu t}$,

$$\begin{aligned} u(t) &= \arg \max_{u \in \mathbb{R}_{\geq 0}} e^{\nu t} (I_* - I_0(t))(-\nu x + u) + g(t)(x^0(t))^\sigma - h(t)\frac{b}{2}u^2 = \\ \arg \max_{u \in \mathbb{R}_{\geq 0}} e^{\nu t} (I_* - I_0(t))u - h(t)\frac{b}{2}u^2 &= \frac{e^{\nu t}}{bh(t)}(I_* - I_0(t)) \quad \text{for a.a. } t \in \mathbb{T}. \end{aligned}$$

Consider $I(t) \triangleq I_* - I_0(t)$; differentiating $I(t)$ with respect to t , we finally close (4a)–(4b) into the boundary problem

$$\dot{x}^0 = -\nu x^0 + \frac{e^{\nu t}}{bh(t)}I, \quad x^0(0) = K_0, \quad (31a)$$

$$\dot{I} = -\sigma g(t)e^{-\nu t}(x^0)^{\sigma-1}, \quad (31b)$$

$$I(\tau'_n) \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (31c)$$

Each τ' -optimal control generates the unique solution of this problem. For $\sigma = 1$ if such solution exists then there exists a finite limit $\lim_{n \rightarrow \infty} \int_0^{\tau'_n} e^{-\nu t} g(t) dt$ for some $\tau'' \subset \tau'$.

Note that to construct this boundary problem we have to know the subsequence $\tau' \subset \tau$. In terms of the initial sequence τ , it is only possible to claim that, for a solution (x^0, I) of problem (31a)–(31b), $0_{\mathbb{X}}$ is the partial limit of the sequence $(I(\tau_n))_{n \in \mathbb{N}}$. If for some functions g, h for some sequence $\tau \uparrow \infty$ there are multiple τ -solutions, then each of them has its own I and subsequence τ' . Also note that if we do not know the sequence τ , then to find a weakly uniformly overtaking optimal control, we have to solve problem (31a)–(31b) for the boundary condition

$$\liminf_{t \rightarrow \infty} I(t) \leq 0 \leq \limsup_{t \rightarrow \infty} I(t).$$

Now suppose that $g(t) \geq 0, h(t) > 0$ for a.a. $t \in \mathbb{T}$. Then, there exists the common limit I_* of all sequences $(I_0(\tau_n))_{n \in \mathbb{N}}$, and for each weakly overtaking optimal control u^0 there exists the unique solution of problem (31a)–(31b) for the boundary condition

$$I(t) \rightarrow +0 \text{ as } t \rightarrow \infty. \quad (31d)$$

It is possible to find the explicit solution of boundary problem (31a),(31b),(31d) in some specific cases. For example, let the discount factor equal $\frac{1}{(1+t)^{4/3}}$, let the factor of technological advancement be equal to 1. For

$$g(t) = h(t) = \frac{1}{(1+t)^{4/3}}, \nu = 0, \sigma = 1/2, b = \frac{3}{8}, K_0 = 1$$

we have

$$x^0(t) = (1+t)^{4/3}, u^0(t) = \frac{4}{3}(1+t)^{1/3}, I(t) = \frac{1}{2(1+t)}, J^{u^0}(t) = (1+t)^{2/3}.$$

The discount factor $g(t) = \frac{1}{(1+t)^{4/3}}$ here is not arbitrary, its power $\alpha = 3,96/2,94 \approx 4/3$ was determined by means of statistic analysis in [48]. A thorough discussion of various discount functions and their properties could be found, for example, in [23, 24, 49]. These papers do not generally assume the discount function to be dominated by a decreasing exponential function and do not assume its monotonicity.

7 Appendix

The proof of Proposition 1. For the sake of brevity, let us denote $\tilde{H} \triangleq \prod_{n \in \mathbb{N}} \tilde{\mathfrak{U}}_n$, and let us equip it with Tikhonov topology. Let $\tilde{\Delta} : \tilde{\mathfrak{U}} \rightarrow \tilde{H}$ be given by $\tilde{\Delta}(\eta) \triangleq (\tilde{\pi}_n(\eta))_{n \in \mathbb{N}}$ for all $\eta \in \tilde{\mathfrak{U}}$. It is a homeomorphism by continuity of the maps $\tilde{\pi}_n$ and $\tilde{\pi}_n \circ \tilde{\Delta}^{-1}$.

Let $n, k \in \mathbb{N}, (n > k)$. Then, the space $\tilde{\mathfrak{U}}_n$ is included in $\tilde{\mathfrak{U}}_k$ by the mapping $\tilde{\pi}_k^n(\eta) \triangleq \eta|_{[0, k]}$ for all $\eta \in \tilde{\mathfrak{U}}_n$. By $\tilde{\pi}_k^n \circ \tilde{\pi}_i^k = \tilde{\pi}_i^n$ for all $n, k, i \in \mathbb{N}, (n > k > i)$, we have the projective sequence of the topological spaces $\{\tilde{\mathfrak{U}}_n, \tilde{\pi}_k^n\}$; and we can define the inverse limit [26, III.1.5], [25, 2.5.1]. In our notation, we can write it in the form $\varprojlim \{\tilde{\mathfrak{U}}_n, \tilde{\pi}_k^n\} \triangleq \tilde{\Delta}(\tilde{\mathfrak{U}}) \subset \tilde{H}$. As shown above, $\tilde{\Delta}$ is a homeomorphism; hence, $\tilde{\mathfrak{U}}$ is homeomorphous to $\tilde{\Delta}(\tilde{\mathfrak{U}})$. Now, by Kurosh Theorem [26, III.1.13], the inverse limit $\tilde{\Delta}(\tilde{\mathfrak{U}})$ of compacts $\tilde{\mathfrak{U}}_n$ is compact, and $\tilde{\mathfrak{U}}$ is a compact too. Similarly, from [25, 4.2.5] and [46, IV.3.11] it follows that $\tilde{\mathfrak{U}}$ is also metrizable.

Repeating the reasonings without \sim or referring to [25, 3.4.11] and [25, 2.5.6] yields $\mathfrak{U} \cong \varprojlim \{\mathfrak{U}_n, \pi_k^n\} \triangleq \Delta(\mathfrak{U}) \subset H$.

For each $n \in \mathbb{N}$, let the mapping $e_n : \mathfrak{U}_n \rightarrow \tilde{\mathfrak{U}}_n$ be given by $e_n(u)(t) \triangleq (\tilde{\delta} \circ u)(t) = \tilde{\delta}_{u(t)}$ for all $t \in [0, n], u \in \mathfrak{U}_n$. Since for all $n, k \in \mathbb{N}, n > k$ it holds that $e_k \circ \pi_k^n = e_n$, we have the projective system $\{e_n, \pi_k^n\}$. Passing to the inverse limit, we obtain the mapping $e_\Delta : \Delta(\mathfrak{U}) \rightarrow \tilde{\Delta}(\tilde{\mathfrak{U}})$; from $e_n \circ \pi_n = \tilde{\pi}_n \circ \tilde{\delta}$ we have $e_\Delta \circ \Delta = \tilde{\Delta} \circ \tilde{\delta}$, and from $\tilde{\mathfrak{U}}_n = cl\,e_n(\mathfrak{U}_n)$ ([46]) we have $\tilde{\Delta}(\tilde{\mathfrak{U}}) = cl\,e_\Delta(\Delta(\mathfrak{U})) = cl\,(\tilde{\Delta} \circ \tilde{\delta})(\mathfrak{U})$; now, by continuity of $\tilde{\Delta}^{-1}$, we obtain $\tilde{\mathfrak{U}} = cl\,\tilde{\delta}(\mathfrak{U})$.

The mapping $\tilde{\mathfrak{U}}[\eta]$ is continuous by virtue of, for example, [45, Theorem 3.5.6]; the set $\tilde{\mathfrak{U}}[\eta](\tilde{\mathfrak{U}})$ is compact as a continuous image of a compact. In what follows, is sufficient to use $\tilde{\mathfrak{U}} = cl\,\tilde{\delta}(\mathfrak{U})$.

Replacing a and the compact Ξ with the mapping (f, g) and the compact $\{(x_{**}, 0)\}$, we obtain the continuous dependence on η for the maps $\varphi^\eta, \tilde{J}^\eta$. \square

The proof of Proposition 2. For all $n \in \mathbb{N}$, let us consider set

$$\bar{G}_n \triangleq \left\{ (t, y(t)) \mid \forall y \in \tilde{\mathfrak{U}}[\tilde{u}^0], t \in [0, n] \right\};$$

by the theorem of continuous dependence of solutions on initial conditions this set is compact as a continuous image of a compact Ξ . Therefore, on this set, the function $a(t, y, u^0(t))$ is Lipschitz continuous with respect to y for the certain Lipschitz constant $L_n \triangleq L_{\bar{G}_n}^a \in \mathcal{L}_{loc}^1(\mathbb{T}, \mathbb{T})$. For all $t \in [0, n]$, define $M_n(t) \triangleq \int_0^t L_n(\tau) d\tau$. Note that this function is absolutely continuous and monotonically nondecreasing.

Fix $n \in \mathbb{N}$; for all $t \in [n-1, n], u \in \mathfrak{U}$, let us consider the number

$$R(t, u) \triangleq \sup_{y \in \bar{G}_n} \|a(t, y, u) - a(t, y, u^0(t))\|_E.$$

Here, the norm is continuous with respect to y and u , and y assumes values from the compact set; now, for every $u \in \mathbb{U}$ by [18, Theorem 3.7] the supremum reaches the maximum for the certain function $y_{\max}[u] \in \mathcal{L}^1([n, n-1], \bar{G}_n)$. Hence, $R(t, u)$ is measurable with respect to t for each $u \in \mathbb{U}$.

Fix a $t \in [n-1, n]$; for each sufficiently small neighborhood $\mathcal{Y} \subset \mathbb{U}$, by continuity of $a(t, \cdot, \cdot)$ on compact $\bar{G}_n \times cl \mathcal{Y}$, there exists a function $\omega^t \in C(\mathbb{T}, \mathbb{T})$ such that $\omega^t(0) = 0$ and

$$\left| \left| a(t, y, u') - a(t, y, u^0(t)) \right| - \left| a(t, y, u'') - a(t, y, u^0(t)) \right| \right| < \omega^t(\|u' - u''\|) \quad (32a)$$

holds for every $y \in \bar{G}_n, u', u'' \in \mathcal{Y}$. Without loss of generality, assume $R(t, u') \leq R(t, u'')$. Now, by definition, $R(t, u') \geq \left| a(t, y, u') - a(t, y, u^0(t)) \right|$, and, substituting $y \triangleq y_{\max}(u'')(t)$ into (32a), we obtain $0 \leq R(t, u'') - R(t, u') \leq \omega^t(\|u' - u''\|)$; i.e., R is continuous with respect to the variable u on each sufficiently small neighborhood $\mathcal{Y} \subset \mathbb{U}$; therefore on \mathbb{U} too. Thus, the function $R : [n-1, n] \times \mathbb{U} \rightarrow \mathbb{T}$ is a Carathéodory function.

Let us note that by considering all $n \in \mathbb{N}$, we define the Carathéodory function R on the whole $\mathbb{T} \times \mathbb{U}$. Moreover, by construction, $R(t, u^0(t)) \equiv 0$. Hence, it is correct to define $w^0 \in (Null)(u^0)$ by the rule

$$w^0(t, u) \triangleq \|u - u^0(t)\| + e^{M_n(t)} R(t, u) \quad \forall n \in \mathbb{N}, t \in [n-1, n], u \in \mathbb{U}. \quad (32b)$$

Consider arbitrary $n \in \mathbb{N}$, $\vartheta \in [0, n]$, and $(\vartheta, y_1^*), (\vartheta, y_2^*) \in \bar{G}_n$. There exist solutions $y_1, y_2 \in \tilde{\mathfrak{A}}[\tilde{u}^0]$ of equation (3c), for the initial conditions $y_i(\vartheta) = y_i^*$. Let us introduce functions

$$r(t) \triangleq y_1(t) - y_2(t), \quad W_+(t) \triangleq e^{M_n(t)} \|r(t)\|_E \quad \forall t \in [0, n].$$

By Lipshitz continuity of the right-hand side of (3c) we obtain $\|\dot{r}(t)\|_E \geq -L_n(t)\|r(t)\|_E$, and

$$\frac{dW_+^2(t)}{dt} = 2L_n(t)W_+^2(t) + 2e^{2M_n(t)}r(t)\dot{r}(t) \geq 2L_n(t)W_+^2(t) - 2L_n(t)W_+^2(t) = 0.$$

Thus, the function W_+ is nondecreasing, and finally for all $(\vartheta, y_1^*), (\vartheta, y_2^*) \in \bar{G}_n$ we have

$$\|\mathcal{K}(\vartheta, y_1^*) - \mathcal{K}(\vartheta, y_2^*)\|_E = W_+(0) \leq W_+(\vartheta) = e^{M_n(\vartheta)} \|y_1^* - y_2^*\|_E. \quad (32c)$$

Let us now consider $\eta \in \tilde{\mathfrak{U}}$, $y \in \tilde{\mathfrak{A}}[\eta]$, $T \in \mathbb{T}$ such that $\mathcal{K}(\vartheta, y(\vartheta)) \in \Xi$ for all $\vartheta \in [0, T]$. Fix arbitrary $n \in \mathbb{N}$ and $\vartheta_1, \vartheta_2 \in [0, T] \cap [n-1, n]$, $\vartheta_1 < \vartheta_2$ there exists the solution $y^0 \in \tilde{\mathfrak{A}}[\tilde{u}^0]$ such that $y^0(\vartheta_1) = y(\vartheta_1)$. By construction of \bar{G}_n , we have $(t, y(t)), (t, y^0(t)) \in \bar{G}_n$ for all $t \in [\vartheta_1, \vartheta_2]$. Let us also define

$$r \triangleq y^0(t) - y(t), \quad W_-(t) \triangleq e^{-M_n(t)} \|r(t)\|_E \quad \forall t \in [\vartheta_1, \vartheta_2].$$

Then $W_-(\vartheta_1) = 0$. Now,

$$\begin{aligned} \frac{dW_-^2(t)}{dt} &= 2e^{-2M_n(t)} r(t)\dot{r}(t) - 2L_n(t)W_-^2(t) = \\ &= 2e^{-2M_n(t)} r(t) (\dot{y}^0(t) - a(t, y(t), u^0(t)) + a(t, y(t), u^0(t)) - \dot{y}(t)) - 2L_n(t)W_-^2(t) \leq \\ &= 2e^{-2M_n(t)} \|r(t)\|_E \int_{U(t)} R(t, u) d\eta(t) + 2L_n(t)W_-^2(t) - 2L_n(t)W_-^2(t) \leq \\ &= 2e^{-M_n(t)} W_-(t) \int_{U(t)} R(t, u) d\eta(t) \leq 2e^{-2M_n(t)} W_-(t) \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt}. \end{aligned}$$

Since function W_- is nonnegative, for a. a. $t \in \{t \in [\vartheta_1, \vartheta_2] \mid W_-(t) \neq 0\}$ we obtain

$$\frac{dW_-(t)}{dt} \leq e^{-2M_n(t)} \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt} \leq e^{-2M_n(\vartheta_1)} \frac{d\mathfrak{L}_{w^0}[\eta](t)}{dt}.$$

This inequality is trivial for $t \in [\vartheta_1, \vartheta_2], t < \sup\{t \in [\vartheta_1, \vartheta_2] \mid W_-(t) = 0\}$; whence, integrating inequality in $t \in [\vartheta_1, \vartheta_2]$, we obtain

$$W_-(\vartheta_2) = W_-(\vartheta_2) - W_-(\vartheta_1) \leq e^{-2M_n(\vartheta_1)} (\mathfrak{L}_{w^0}[\eta](\vartheta_2) - \mathfrak{L}_{w^0}[\eta](\vartheta_1)).$$

But $\varkappa(\vartheta_2, y^0(\vartheta_2)) = y^0(0) = \varkappa(\vartheta_1, y^0(\vartheta_1)) = \varkappa(\vartheta_1, y(\vartheta_1))$, hence, we have

$$\begin{aligned} \|\varkappa(\vartheta_2, y(\vartheta_2)) - \varkappa(\vartheta_1, y(\vartheta_1))\|_E &= \|\varkappa(\vartheta_2, y^0(\vartheta_2)) - \varkappa(\vartheta_2, y(\vartheta_2))\|_E \stackrel{(32c)}{\leq} \\ &e^{M_n(\vartheta_2)} \|y^0(\vartheta_2) - y(\vartheta_2)\|_E = e^{2M_n(\vartheta_2)} W_-(\vartheta_2) \leq \\ &e^{2M_n(\vartheta_2) - 2M_n(\vartheta_1)} (\mathfrak{L}_{w^0}[\eta](\vartheta_2) - \mathfrak{L}_{w^0}[\eta](\vartheta_1)). \end{aligned} \quad (32d)$$

Fix arbitrary $t \in [0, T]$. For each $\varepsilon > 0$ we can split interval $[0, t]$ into the intervals of the form $[\vartheta', \vartheta'']$ such that $M_n(\vartheta'') - M_n(\vartheta') < \varepsilon$ and $[\vartheta', \vartheta''] \subset [n-1, n]$ for the certain $n \in \mathbb{N}$. But, (32d) holds for every interval, i.e.,

$$\|\varkappa(\vartheta'', y(\vartheta'')) - \varkappa(\vartheta', y(\vartheta'))\|_E \leq e^{2\varepsilon} (\mathfrak{L}_{w^0}[\eta](\vartheta'') - \mathfrak{L}_{w^0}[\eta](\vartheta')).$$

Summing for all intervals, by $\varkappa(0, y(0)) = y(0)$ and by the triangle inequality, we obtain $\|\varkappa(t, y(t)) - y(0)\|_E \leq e^{2\varepsilon} \mathfrak{L}_{w^0}[\eta](t)$ for every $t \in [0, T]$. Arbitrariness of $\varepsilon > 0$ completes the proof of the Proposition 2. \square

The proof of Proposition 3. For every $n \in \mathbb{N}$, let us consider the problem

$$J^n(\tau_n) - \gamma_n \mathfrak{L}_w[\eta](\tau_n) = \int_0^{\tau_n} \int_{U(t)} g(t, x^n(t), u) d\eta(t) dt - \gamma_n \mathfrak{L}_w[\eta](\tau_n) \rightarrow \max.$$

Here, the functional is bounded from above by the number $J^{u^0}(\tau_n) + \gamma_n^2$, therefore, it has the supremum. Every summand continuously depends on η , which covers the compact $\widetilde{\mathfrak{M}}$; therefore, there is an optimal solution for this problem in $\widetilde{\mathfrak{M}}$; let us denote one of them by η^n , and its trajectory by x^n .

For every $\gamma \in \mathbb{T}$ let the function $\mathcal{H}_\gamma : \mathbb{X} \times \mathbb{T} \times \mathbb{U} \times \mathbb{T} \times \mathbb{X} \rightarrow \mathbb{R}$ be given by

$$\mathcal{H}_\gamma(x, t, u, \lambda, \psi) \triangleq \mathcal{H}(x, t, u, \lambda, \psi) - \gamma w(t, u).$$

Then, by the Maximum Principle [17, Theorem 5.2.1], there exists $(\lambda^n, \psi^n) \in (0, 1] \times C([0, n], \mathbb{X})$ such that relation (5a) and the transversality condition $\psi^n(\tau_n) = 0$ hold, and

$$\begin{aligned} \sup_{p \in U(t)} \mathcal{H}_{\gamma_n}(x^n(t), t, p, \lambda^n, \psi^n(t)) &= \int_{U(t)} \mathcal{H}_{\gamma_n}(x^n(t), t, u, \lambda^n, \psi^n(t)) d\eta^n(t), \\ \dot{\psi}^n(t) &= - \int_{U(t)} \frac{\partial \mathcal{H}_{\gamma_n}}{\partial x}(x^n(t), t, u, \lambda^n, \psi^n(t)) d\eta^n(t) \end{aligned} \quad (33a)$$

also hold for a.a. $t \in [0, \tau_n]$.

Let us extend the $(x^n, \eta^n, \lambda^n, \psi^n)$ to $[\tau_n, \infty)$ by the generalized control $\widetilde{u}^0|_{[\tau_n, \infty)}$. Let us denote by \mathfrak{Z}^n the set of (x, u, λ, ψ) that satisfy relations (5a), $(4a)-(4b)$ a. e. on \mathbb{T} , satisfy relation (33a) a. e. on $[0, \tau_n]$, and possess the property $\widetilde{u}^0|_{[\tau_n, \infty)} = \eta^n|_{[\tau_n, \infty)}$. Now we have $(x^n, \eta^n, \lambda^n, \psi^n) \in \mathfrak{Z}^n$ for every $n \in \mathbb{N}$.

Let us note that all \mathfrak{Z}^n are closed and, since these sets are contained in the compact $\tilde{\mathfrak{Y}}$, these sets are also compact. Hence, the sequence $(x^n, \eta^n, \lambda^n, \psi^n)_{n \in \mathbb{N}}$ has the limit point $(x^{00}, \eta^0, \lambda^0, \psi^0) \in \tilde{\mathfrak{Y}}$. Considering, if need be, the subsequence, we can assume that it is the limit of the sequence itself.

For all $t, \gamma, \lambda \in \mathbb{T}$, $(x, \psi) \in \mathbb{X} \times \mathbb{X}$, denote by $\mathcal{P}_{\gamma, \lambda}(t; x, \psi)$ the set of $p \in U(t)$ that realize the maximum of $\mathcal{H}_{\gamma}(x, t, p, \lambda, \psi)$. For all $\gamma, \lambda \in \mathbb{T}$, $(x, \psi) \in \mathbb{X} \times \mathbb{X}$, the compact-valued map $t \mapsto \mathcal{P}_{\gamma, \lambda}(t; x, \psi)$ has a measurable selector by virtue of [18, Theorem 3.7]. Then, by [45, Lemm 2.3.11], for an arbitrary function $(x, \psi) \in C(\mathbb{T}, \mathbb{X} \times \mathbb{X})$ the map $t \mapsto \mathcal{P}_{\gamma, \lambda}(t; (x, \psi)(t))$ also has a measurable selector. Therefore, since relation (33a) also depends on x, ψ and on the parameters γ and λ upper semicontinuously, and all the relations are integrally bounded on bounded sets, by virtue of [45, Theorem 3.5.6], on each finite interval for the funnels of solutions of (4a)–(4b), (33a), we have upper semicontinuity by γ, λ . In particular, since $\gamma_n \rightarrow 0$ and $\lambda^n \rightarrow \lambda^0$, the upper limit of the compacts \mathfrak{Z}^n is included in $\tilde{\mathfrak{Z}}$. Hence, $(x^{00}, \eta^0, \lambda^0, \psi^0) \in \tilde{\mathfrak{Z}}$.

On the other side, by $w \in (Null)(u^0)$ and by optimality of η^n, u^0 for their problems, we obtain

$$\tilde{J}^{\eta^n}(\tau_n) - \gamma_n \mathfrak{L}_w[\eta^n](\tau_n) \geq J^{u^0}(\tau_n) \stackrel{(2)}{\geq} \tilde{J}^{\eta^n}(\tau_n) - \gamma_n^2$$

therefore, we have $\gamma_n \mathfrak{L}_w[\eta^n](\tau_n) \leq \gamma_n^2$. By $\tilde{u}^0|_{[\tau_n, \infty)} = \eta^n|_{[\tau_n, \infty)}$, we obtain

$$\mathfrak{L}_w[\eta^n](t) \leq \gamma_n \quad \forall t \in \mathbb{T}. \quad (33b)$$

For each $t \in \mathbb{T}$, passing to the limit as $n \rightarrow \infty$, we obtain that $\mathfrak{L}_w[\eta^0] \leq 0$; i.e., $\mathfrak{L}_w[\eta^0](t) = 0$ for all $t \in \mathbb{T}$. Since $w \in (Null)(u^0)$, we have $\eta^0 = \tilde{u}^0$ a.e. on \mathbb{T} , hence $x^{00} \equiv x^0$ and $(x^0, u^0, \lambda^0, \psi^0) \in \mathfrak{Z}$. Moreover, from (33b), we have $\|\mathfrak{L}_w[\eta^n]\|_C \rightarrow 0$. \square

Acknowledgements I am grateful to an anonymous referee for helpful comments. I would like to express my gratitude to S. M. Aseev, A. G. Chentsov, A. M. Tarasyev, N. Yu. Lukoyanov, and Yu. V. Averboukh for valuable discussion in course of writing this article. Special thanks to Ya. V. Salii for the translation.

This supported by the Russian Foundation for Basic Research (RFBR) under grant No 12-01-00537.

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